Non-perturbative quantization of the electroweak model's electrodynamic sector

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Consider the Euclidean functional integral representation of any physical process in the electroweak model. Integrating out the fermion degrees of freedom introduces twenty-four fermion determinants. These multiply the Gaussian functional measures of the Maxwell, Z, W and Higgs fields to give an effective functional measure. Suppose the functional integral over the Maxwell field is attempted first. This paper is concerned with the large amplitude behavior of the Maxwell effective measure. It is assumed that the large amplitude variation of this measure is insensitive to the presence of the Z, W and H fields; they are assumed to be a subdominant perturbation of the large amplitude Maxwell sector. Accordingly, we need only examine the large amplitude variation of a single QED fermion determinant. To facilitate this the Schwinger proper time representation of this determinant is decomposed into a sum of three terms. The advantage of this is that the separate terms can be non-perturbatively estimated for a measurable class of large amplitude random fields in four dimensions. It is found that the QED fermion determinant grows faster than $\exp\left[ce^2\int \mathrm{d}^4x\, F_{\mu\nu}^2\right]$, $c>0$, in the absence of zero mode supporting random background potentials. This raises doubt on whether the QED fermion determinant is integrable with any Gaussian measure whose support does not include zero mode supporting potentials.

Including zero mode supporting background potentials can result in a decaying exponential growth of the fermion determinant. This is *prima facie* evidence that Maxwellian zero modes are necessary for the non-perturbative quantization of QED and, by implication, for the non-perturbative quantization of the electroweak model.

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I. INTRODUCTION

It is not known if the electroweak model can be non-perturbatively quantized. This requires the convergence of the unexpanded functional integrals over all classical field configurations for the vacuum expectation values of its field operators. It is assumed that the integrals have been continued to Euclidean space to make mathematical sense out of them and that ultraviolet and volume cutoffs are in place in their integrands. Their introduction will be discussed later. Since the quantization is non-perturbative most of the functional integrals cannot be done explicitly. Therefore, the criteria for the non-perturbative renormalization of the model are not known *ab initio*. Immediately one is confronted with an external field problem: do the regulated integrands grow slowly enough with large amplitude field variations for the functional integrals to converge? It is the aim of this paper to examine this minimal requirement for the non-perturbative quantization of the electroweak model.

Presumably the order of doing the functional integrals is irrelevant aside from their technical difficulty. If so, it is reasonable to begin with what is well-known. Accordingly, we first integrate out the fermions. Then the answer to the above question partly depends on knowing the strong field behavior of each of the 6 lepton and 3×6 quark determinants obtained by integrating out the three generations of leptons and quarks, including their three colors. For example, the electron and its associated neutrino field[1] contribute the following factor to the Euclidean functional integral representation of any electroweak process after spontaneous symmetry breaking:

$$\begin{aligned}&\det\left[\not{P}+m_e+e\not{A}+\frac{g}{2\cos\theta_W}\not{Z}\left(\frac{1-\gamma_5}{2}\right)-\frac{g\sin^2\theta_W}{\cos\theta_W}\not{Z}+\frac{gm_e}{2M_W}H\right]\times\\&\det\left[\not{P}-\frac{g}{2\cos\theta_W}\not{Z}\left(\frac{1-\gamma_5}{2}\right)-\frac{g^2}{2}\not{W}^+\left(\frac{1-\gamma_5}{2}\right)S_e\not{W}^-\left(\frac{1-\gamma_5}{2}\right)\right].\end{aligned}\tag{1.1}$$

Here A_μ, Z_μ, $W_\mu^\pm$, and H are the Maxwell, neutral and charged vector boson and Higgs fields; S_e, the inverse of the operator in brackets in the first determinant, is the electron propagator in the presence of the A, Z and H

[1] The extension of the model to massive neutrinos and their mixing is not considered here as it will not affect the main results of this paper.

fields; m_e and M_W are the electron and W-boson masses; e is the positron electric charge; θ_W is the Weinberg angle and $g = e/\sin\theta_W$. The result in (1.1) follows by inspection of the electroweak Lagrangian [1] and an elementary integration over the electroweak action quadratic in the fermion fields [2]. The twenty-four determinants multiply the Gaussian measures $d\mu(A)\,d\mu(Z)\,d\mu(W)\,d\mu(H)$ as does the remainder of the electroweak action denoted by $\exp\left[-\int \mathrm{d}^4x\,\mathcal{L}(A, Z, W^{\pm}, H)\right]$. Considering the complexity of the Feynman rules in the 't Hooft-Feynman gauge a non-perturbative calculation may simplify in the unitary gauge. The absence of the Goldstone bosons $\chi, \varphi^{\pm}$ in the determinants in (1.1) indicates that this gauge has been selected.

An ultraviolet cutoff has to be introduced into the A, Z, W and H field propagators. As these fields are to be integrated over they are assumed to be tempered distributions. In order to calculate the fermion determinants these fields need to be smoothed following the procedure outlined at the beginning of Sec. VII for QED. The smoothing procedure introduces an ultraviolet cutoff in the associated propagators when calculating the fields' covariances with the above Gaussian gauge-fixed measures as in Eq.(7.2). Thus the ultraviolet cutoffs are introduced by functionally integrating the electroweak model.

The fermion determinants contain all fermion loops and hence the anomalies. The process for cancelling them in this paper begins by noting that the determinants, such as those in (1.1), are ill-defined as they stand. Mathematical sense can be made of them by subtracting out all loops whose degree of divergence is 2, 1 and 0. The subtraction process is illustrated by (F1) in Appendix F for the case of QED. As a representative example consider the γW^+W^- triangle graph containing three fermion propagators. Schematically the electron neutrino determinant in (1.1) is subtracted so that det$\to$ $\exp[\Pi(ee\nu_e)$ + other subtractions$]$ $\times$ $\det_R$, where $\det_R$ is a well-defined remainder determinant similar to $\det_5$ in (F1) and (F2); $\Pi(ee\nu_e)$ denotes the first generation lepton triangle graph for $\gamma \to W^+W^-$. When the 23 remaining determinants are subtracted the exponentiated subtractions combine to give the following result for the sum of all the graphs contributing to the first generation γW^+W^- triangle anomaly:

$$\begin{aligned}\exp\{\Pi(ee\nu_e) &+ 3[\Pi(ddu) + \Pi(uud)]|V_{ud}|^2 \\ &+ 3[\Pi(ssu) + \Pi(uus)]|V_{us}|^2 \\ &+ 3[\Pi(bbu) + \Pi(uub)]|V_{ub}|^2 \\ &+ \text{other subtractions}\} \times \Pi_{i=1}^{24}\det{}_{R_i}.\end{aligned} \tag{1.2}$$

Here u, d, s, b refer to quark flavors and V_{ij} is the CKM quark mixing matrix [1]. The anomaly is removed by subtracting out the zero-mass limit of these graphs which we denote by Π_0 . Then the anomaly bearing graphs reduce to

$$\exp\{\Pi_0(ee\nu_e)+3[\Pi_0(uud)+\Pi_0(ddu)](|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2)\} \tag{1.3}$$

since there is no difference between the free u,d,s and b propagators in the massless limit. Noting that the unitarity of the CKM matrix requires the sum of the matrix elements in (1.3) to be one, the sum of the color weighted γ -vertices in (1.3) results in the cancellation of the first generation γW^+W^- triangle anomaly. This procedure can be continued until all of the three and four leg anomalies in the three generations cancel as they are known to do. These determinant regularizations should be done before they are inserted into the functional integrals over the gauge and Higgs fields.

Summarizing, it is necessary to define the fermion determinants by removing their ill-defined loops by making subtractions that are then either renormalized or cancelled among themselves. This happens to lead to anomaly cancellation at the three and four external leg level. Of course it has not been proved that the product of the remainder determinants is free of terms that can block the non-perturbative renormalization of the electroweak model [55].

It is known that when $\Pi_{i=1}^{24}\det_{R_i}$ is loop-expanded it contains an exponentiated sum of absolutely convergent graphs beginning with the pentagon graph. These can be calculated in a manifestly gauge invariant way and cannot contain anomalies. The fact that the perturbative expansion of $\Pi_{i=1}^{24}\det_{R_i}$ is anomaly-free leaves open the possibility that this determinant product may eventually be shown to be part of a non-perturbative, anomaly-free, gauge preserving regularization of the electroweak model.

Assuming the functional integrals converge the process of renormalization follows next with the introduction of counterterms to remove the regulators. Presumably the result is in terms of the physical parameters e, M_W, M_Z, M_H, m_i -the charged fermion masses- and the renormalized quark mixing matrix V_{ij} after continuing from an intermediate renormalization scheme in Euclidean space to on-shell renormalization in Minkowski space.

The observation that $\mathcal{L}$ is no more than quadratic in A_μ, that A_μ does not couple directly to H, that a considerable amount is known about the QED determinant $\det(\not{P} - e\not{A} + m)$, and that the regularization of the electrodynamic sector is straightforward suggests that the next simplest functional integration should be over the Maxwell field. Supppose this is decided. Twenty-one of the twenty-four fermion determinants involve the Maxwell field as it appears in the electron's determinant in (1.1) with different charges. Should their combined large amplitude A-field variation increase faster than $\exp\left[ce^2\int \mathrm{d}^4x\,F_{\mu\nu}^2\right]$, $c > 0$ then the integration over the Maxwell field with any Gaussian measure would be divergent, and the non-perturbative quantization of the electroweak model would be doubtful. The $F_{\mu\nu}$-dependence is expected since the determinants are gauge invariant.

It is assumed that the strong Maxwell field behavior of these determinants can be obtained by decoupling them from the electroweak model by setting $g = 0$. Future theorems dealing with the assumed sub-dominant growth of the remainder determinants can and should be produced. Noting this, there remains a product of twenty one determinants of the form $\det(\not{P} - q\not{A} + m)$ so that we need only calculate one of them. Accordingly, this paper considers the the non-perturbative quantization of the electroweak model's electrodynamic sector. It is found that this can be done only under restrictive conditions. If the sub-dominance of the remainder determinants assumed here is valid then these conditions extend to the complete electroweak model.

II. PRELIMINARIES

Confining attention to QED, sense has to be made of the infinite dimensional determinant $\det(\not{P} - e\not{A} + m)$, where $e > 0$ from here on. It is first normalized to one when $e = 0$ by dividing it by $\det(\not{P} + m)$ to get $\det(1 - eS\not{A})$, where S is the free electron propagator. To make this well-defined it has to be regularized and made ultraviolet finite by a second order charge renormalization subtraction. A representation of the regulated and renormalized determinant, denoted by $\det_{\mathrm{ren}}$, is given by Schwinger's proper time definition [3]

$$\ln \det{}_{\mathrm{ren}}(1 - e_0 S\not{A}) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\left(\mathrm{Tr}\left\{e^{-P^2 t} - \exp\left[-\left(D^2 + \frac{e_{\mathrm{o}}}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)t\right]\right\} + \frac{e_{\mathrm{o}}^2\|F\|^2}{24\pi^2}\right)e^{-tm_{\mathrm{o}}^2}, \tag{2.1}$$

where $D_\mu = P_\mu - e_{\mathrm{o}}A_\mu$, $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i$, $\gamma_\mu^\dagger = -\gamma_\mu$, $\|F\|^2 = \int \mathrm{d}^4x\, F_{\mu\nu}^2$, and e_{o}, m_{o} are the unrenormalized charge and mass. The last term in (2.1) results in a second-order charge renormalization subtraction in the one-particle irreducible photon self-energy $\Pi(k^2)$ at zero momentum transfer as in Eq.(C7), Appendix C. Therefore, as long as A_μ remains a classical field e_{o} and m_{o} are the physical parameters e and m. Quantizing A_μ by integrating over it will require a further charge renormalization subtraction given by $1/e_{\mathrm{o}}^2 = 1/e^2 + \Pi(0, e_{\mathrm{o}}^2 D_{\mathrm{o}})$, where $\Pi(0, e_{\mathrm{o}}^2 D_{\mathrm{o}})$ is the 1PI photon self-energy at $k^2 = 0$ with the one-loop contribution omitted. It is a functional of the exact unrenormalized photon propagator D_{o} with $\Pi(0,0) = 0$; it is made finite by the regularization procedure outlined in Sec. VII. As renormalization will not be considered further the subscript o will be dropped in (2.1) with the understanding that e and m are the *unrenormalized* charge and mass in what follows.

Having defined $\det_{\mathrm{ren}}$ the effective measure for the Maxwell field integration is

$$\mathrm{d}\mu(A) = Z^{-1}\mathrm{d}\mu_0(A)\det{}_{\mathrm{ren}}(1 - eS\not{A}) \tag{2.2}$$

where the gauge-fixed Gaussian measure for the random potential A_μ is now denoted by $d\mu_0$. It has mean zero and covariance

$$\int \mathrm{d}\mu_0\, A_\mu(x)A_\nu(y) = D_{\mu\nu}(x - y), \tag{2.3}$$

where $D_{\mu\nu}$ is the photon propagator in a fixed gauge. The vacuum-vacuum amplitude Z in (2.3) is

$$Z = \int \mathrm{d}\mu_0\, \det{}_{\mathrm{ren}}, \tag{2.4}$$

so that $\int \mathrm{d}\mu(A) = 1$. The measure (2.2) appears in the non-perturbative calculation of every physical process in QED such as the Euclidean Green function for $2n$ external fermions and m photons,

$$\begin{aligned} &S_{\mu_1\ldots\mu_m}(x_1,\ldots,x_n;y_1,\ldots,y_n;z_1,\ldots,z_m) \\ &\quad = Z^{-1}\int \mathrm{d}\mu_0(A)\, \det{}_{\mathrm{ren}}(1 - eS\not{A})\det\left[S(x_i,y_j|eA)\right]_{i,j=1}^n \prod_{k=1}^m A_{\mu_k}(z_k), \end{aligned} \tag{2.5}$$

where $S(x,y|eA)$ is the electron propagator in the external potential A_μ.

Any attempt to calculate the integrals in (2.4) and (2.5) will encounter ultraviolet divergences that require

regularization. How this regularization is introduced will be discussed in Sec. VII. In addition Z requires a volume cutoff that will be discussed in in Sec. VII as well. A volume cutoff enters QED solely by its determinant to render the vacuum energy finite when the determinant is integrated. Assuming that the functional integrations in (2.4) and (2.5) converge, there remains the task of removing the ultraviolet regulator and volume cutoff by some as yet unknown non-perturbative renormalization procedure that preserves the unitarity of S-matrix elements. The difficulty of implementing this procedure cannot be overstated.

Whether the functional integrals in (2.4) and (2.5) converge depends on $\det_{\rm ren}$'s behavior for large amplitude variations of a measurable set of random fields $F_{\mu\nu}$ on $\mathbb{R}^4$. Since e always multiplies $F_{\mu\nu}$ it will be sufficient to consider the strong coupling behavior of $\det_{\rm ren}$.

This leads to one of the main results of this paper. Although (2.1) is compact and intuitive it – and all other representations – have so far failed to give any explicit information on the strong coupling behavior of $\det_{\rm ren}$ for random fields on $\mathbb{R}^4$. To remedy this an exact representation of $\ln\det_{\rm ren}$ is derived from (2.1) that facilitates its strong coupling analysis. Noting that in Euclidean space $F_{\mu\nu}$ may be regarded as a static, four-dimensional magnetic field, the new representation breaks $\ln\det_{\rm ren}$ into a sum of three terms that expose its competing magnetic properties, namely,

$$\begin{aligned}\ln\det{}_{\rm ren} = {} & \text{diamagnetism} + \text{paramagnetism} \\ & + \text{charge renormalization}.\end{aligned} \tag{2.6}$$

The advantage of representation (2.6) of $\det_{\rm ren}$ is that the strong coupling analysis of its separate terms is far easier than their combined form in (2.1). The derivation of (2.6) is given in Sec. III. Suffice it to say here that the sum of the diamagnetic term (Sec.IV) and charge renormalization term (Sec.VI) contribute to $\det_{\rm ren}$'s strong coupling growth while the paramagnetic term (Sec.V) slows it down. Therefore, the non-perturbative quantization of QED critically depends on the paramagnetic term and the class of background fields on which it depends. *Prima facie* evidence is given that zero mode supporting background fields are necessary for the non-perturbative quantization of QED. The presence of substantial numbers of zero modes in the lattice functional integration of QED in its chirally broken phase has been noted [4, 5]. Our result and this observation suggest that Maxwellian zero modes will play a key role in deciding whether the electroweak model can be non-perturbatively quantized. Our conclusions are summarized in Secs.VIC and VIII, and the appendices deal with mathematical details.

III. REPRESENTATION OF $\det_{\rm ren}$

The objective is to obtain an expression for $\det_{\rm ren}$ that manifests the interplay of diamagnetism, paramagnetism and charge renormalization in its strong coupling behavior for random, static, four-dimensional magnetic fields. Rewrite (2.1) as

$$\begin{aligned}&\ln\det{}_{\rm ren}\\ &= \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\, e^{-tm^2}\left[4\mathrm{Tr}\left(e^{-P^2t}-e^{-D^2t}\right)-\frac{e^2\|F\|^2}{48\pi^2}+\mathrm{Tr}\left(e^{-D^2t}-\exp\left[-\left(D^2+\frac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)t\right]\right)+\frac{e^2\|F\|^2}{16\pi^2}\right],\end{aligned} \tag{3.1}$$

where the trace over spin was made in the first term to give a factor of 4. Then (3.1) becomes

$$\ln\det{}_{\rm ren} = 2\ln\det{}_{\rm SQED}+\frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\, e^{-tm^2}\left[\mathrm{Tr}\left(e^{-D^2t}-\exp\left[-\left(D^2+\frac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)t\right]\right)+\frac{e^2\|F\|^2}{16\pi^2}\right], \tag{3.2}$$

where $\ln\det_{\rm SQED}$ is the proper time definition of the formal scalar QED determinant $\ln\det\left\{\left[(P-eA)^2+m^2\right]/(P^2+m^2)\right\}$ with on-shell charge renormalization:

$$\begin{aligned}&\ln\det{}_{\rm SQED}\\ &= \int_0^\infty \frac{\mathrm{d}t}{t}\left[\mathrm{Tr}\left(e^{-P^2t}-e^{-D^2t}\right)-\frac{e^2\|F\|^2}{192\pi^2}\right]e^{-tm^2},\end{aligned} \tag{3.3}$$

Alternatively, $\ln \det_{\rm SQED} = -S_{\rm SQED}$, where $S_{\rm SQED}$ is the one-loop effective action of scalar QED.

Now consider the remaining terms in (3.2) and use the operator identity

$$e^{-t\left(D^2+\frac{1}{2}e\sigma F\right)} - e^{-tD^2} = -\int_0^t \mathrm{d}s\, e^{-(t-s)\left(D^2+\frac{1}{2}e\sigma F\right)}\frac{1}{2}e\sigma F e^{-sD^2}. \tag{3.4}$$

A derivation of (3.4) is given in [6]. Iterating it twice gives

$$\begin{aligned} &e^{-t\left(D^2+\frac{1}{2}e\sigma F\right)} - e^{-tD^2} \\ &= -\int_0^t \mathrm{d}s\, e^{-(t-s)D^2}\frac{1}{2}e\sigma F e^{-sD^2} \\ &\quad + \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2\, e^{-(t-s_1-s_2)D^2}\frac{1}{2}e\sigma F e^{-s_2D^2}\frac{1}{2}e\sigma F e^{-s_1D^2} \\ &\quad - \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 \int_0^{t-s_1-s_2} \mathrm{d}s_3\, e^{-(t-s_1-s_2-s_3)\left(D^2+\frac{1}{2}e\sigma F\right)}\frac{1}{2}e\sigma F e^{-s_3D^2}\frac{1}{2}e\sigma F e^{-s_2D^2}\frac{1}{2}e\sigma F e^{-s_1D^2}. \end{aligned} \tag{3.5}$$

Define the determinant $\det_3$ by

$$\begin{aligned} \ln \det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right) &= \int_0^\infty \frac{\mathrm{d}t}{t}\, e^{-tm^2}\mathrm{Tr}\bigg(\int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 \int_0^{t-s_1-s_2} \mathrm{d}s_3 \\ &\quad \times e^{-(t-s_1-s_2-s_3)\left(D^2+\frac{1}{2}e\sigma F\right)}\frac{1}{2}e\sigma F e^{-s_3D^2}\frac{1}{2}e\sigma F e^{-s_2D^2}\frac{1}{2}e\sigma F e^{-s_1D^2}\bigg), \end{aligned} \tag{3.6}$$

where $\Delta_A^{1/2} = (D^2+m^2)^{-1/2}$. Before proceeding with the derivation of (2.6) it is important to explain what the left-hand side of (3.6) means [7–11].

Thus $\det_3$ is the regularized determinant defined by

$$\det{}_3(1+T) = \det\left[(1+T)\exp\left(-T+\frac{1}{2}T^2\right)\right], \tag{3.7}$$

provided $T \in \mathscr{I}_3$. The trace ideal $\mathscr{I}_p$ $(1 \le p < \infty)$ is defined as those compact operators T with $\|T\|_p^p = \mathrm{Tr}((T^\dagger T)^{p/2}) < \infty$ [8–10]. Because T is compact its eigenvalues are discrete and have finite multiplicity. Therefore, the left-hand side of (3.6) requires that the operator $\Delta_A^{1/2}\sigma F\Delta_A^{1/2} \in \mathscr{I}_3$. This is shown in Appendix A for $F_{\mu\nu} \in \cap_{p>2}L^p(\mathbb{R}^4)$ and $m \neq 0$. Note that this allows zero mode supporting potentials $A_\mu(x)$ with their necessary $1/|x|$ fall off for $|x| \to \infty$. The equivalence of the two sides of (3.6) follows from Theorem 7.2 in [7] where an outline of its proof is given. Because of the inaccessibility of [7] and the importance of $\det_3$ to this paper a proof is given in Appendix B. More will be said about $\det_3$ in Sec. V. But already we anticipate that its presence in $\det_{\rm ren}$ will be a calculational advantage as it deals with a self-adjoint operator acting on countable, square-integrable eigenstates. Put differently, $\det_3$'s calculation reduces to a manageable quantum mechanical problem on bound state energy levels as discussed in Sec. VB.

Continuing with the derivation of (2.6), insert (3.5) and (3.6) in (3.2) to obtain

$$
\begin{aligned}
\ln\det{}_{\rm ren} = 2\ln\det{}_{\rm SQED} + \frac{1}{2}\ln\det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right) \\
+\frac{e^2}{8}\int_0^\infty\frac{\mathrm{d}t}{t}\,e^{-tm^2}\left(\frac{1}{4\pi^2}\|F\|^2-\mathrm{Tr}\int_0^t\mathrm{d}s_1\int_0^{t-s_1}\mathrm{d}s_2 e^{-(t-s_1-s_2)D^2}\sigma F e^{-s_2D^2}\sigma F e^{-s_1D^2}\right).
\end{aligned}
\tag{3.8}
$$

It is shown in Appendix C that the last term in (3.8) can be simplified to give the promised three-term representation of $\ln\det_{\rm ren}$:

$$
\ln\det{}_{\rm ren} = 2\ln\det{}_{\rm SQED} + \frac{1}{2}\ln\det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right) + e^2\int_0^\infty\mathrm{d}t\,e^{-tm^2}\left[\frac{1}{32\pi^2 t}\|F\|^2-\frac{1}{2}\mathrm{Tr}\left(e^{-tD^2}F_{\mu\nu}\Delta_A F_{\mu\nu}\right)\right],
\tag{3.9}
$$

where $\Delta_A = (D^2+m^2)^{-1}$.

Equation (3.9) is equivalent to (2.1), and each term is separately well-defined and gauge invariant. Their order follows that in (2.6). The signs of the first two terms and their connection with diamagnetism and paramagnetism are discussed in the following sections. The last term is connected with charge renormalization and is manifestly positive due to QED's lack of asymptotic freedom.

IV. STRONG COUPLING BEHAVIOR OF $\det_{\rm SQED}$

Let the amplitude of $F_{\mu\nu}(x)$ be set by the parameter $\mathscr{F}$ which has the dimension of L^{-2}. Then break the integral in (3.3) into $\int_0^{1/e\mathscr{F}}$ and $\int_{1/e\mathscr{F}}^\infty$ and use Kato's inequality in the form [12–15]

$$
\mathrm{Tr}\left(e^{-P^2t}-e^{-(P-eA)^2t}\right)\geq 0,
\tag{4.1}
$$

to obtain

$$
\begin{aligned}
&\ln\det{}_{\rm SQED}\geq \\
&\int_0^{1/e\mathscr{F}}\frac{\mathrm{d}t}{t}\left[\mathrm{Tr}\left(e^{-P^2t}-e^{-(P-eA)^2t}\right)-\frac{e^2\|F\|^2}{192\pi^2}\right]e^{-tm^2} \\
&\quad -\frac{e^2\|F\|^2}{192\pi^2}\int_{1/e\mathscr{F}}^\infty\frac{\mathrm{d}t}{t}\,e^{-tm^2}.
\end{aligned}
\tag{4.2}
$$

The inequality in (4.1) reflects the diamagnetism of charged scalar bosons: on average the energy levels of such bosons increase in a magnetic field. This explains the first term in (2.6). The selection of $e\mathscr{F}$ as the scaling parameter is discussed below.

The first integral in (4.2) is dominated by its small-t behavior for $e\gg 1$. Accordingly, make the heat kernel expansion

$$
\begin{aligned}
\mathrm{Tr}\left(e^{-P^2t}-e^{-(P-eA)^2t}\right) = \frac{1}{16\pi^2}\int\mathrm{d}^4x\left[\frac{e^2}{12}F_{\mu\nu}^2+\frac{te^2}{120}F_{\mu\nu}\nabla^2F_{\mu\nu}\right. \\
\left.+\frac{t^2e^2}{1680}F_{\mu\nu}\nabla^4F_{\mu\nu}+\frac{t^2e^4}{1440}\left[({}^\star F_{\mu\nu}F_{\mu\nu})^2-7(F_{\mu\nu}^2)^2\right]\right]+\mathrm{O}(t^3),
\end{aligned}
\tag{4.3}
$$

where ${}^\star F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$. The $\mathrm{O}(F^2)$ terms follow from the result for $\ln\det_{\rm SQED}$ in (C6); the $\mathrm{O}(F^4)$ term is inferred from Schwinger's constant field result for scalar QED [3]

To the author's knowledge there is no proof that QED heat kernel expansions are asymptotic series in t although this is generally assumed. Referring to (4.3) it is evident that continuing the expansion in powers of t requires that $F_{\mu\nu}$ be infinitely differentiable (C^∞). So this is a necessary condition. In Sec.VII we will introduce an ultra-

violet regulator by convoluting the potential A_μ with a function of rapid decrease. The resulting smoothed potential is C^∞. Anticipating Sec. VII we will now assume the fields in (4.3) are C^∞. With this understanding the expansion in (4.3) will now be assumed to be asymptotic so that the truncation error after N terms is

$$\begin{aligned}&\mathrm{Tr}\left(e^{-P^2t}-e^{-(P-eA)^2t}\right)\\&\quad-\sum_{n=0}^{N}a_n(eF)t^n \underset{t\searrow 0}{\sim} a_M(eF)t^M,\end{aligned} \tag{4.4}$$

where a_M is the first nonzero coefficient after a_N [16]. Note that since $[t]=L^2$, the maximum power of $F_{\mu\nu}$ in a_M is $M+2$ so that the truncation error in (4.2) never exceeds $\mathrm{O}(e^2)$.

From (4.3), (4.4) and the result

$$\int_{1/e\mathscr{F}}^{\infty}\frac{\mathrm{d}t}{t}\,e^{-tm^2}=\ln\left(\frac{e\mathscr{F}}{m^2}\right)-\gamma+R, \tag{4.5}$$

where $\gamma=0.5772\ldots$ is Euler's constant and $0<|R|<m^2/(e\mathscr{F})$, obtain from (4.2) for $e\gg 1$

$$\ln\det{}_{\mathrm{SQED}}\geq-\frac{e^2\|F\|^2}{192\pi^2}\ln\left(\frac{e\mathscr{F}}{m^2}\right)+\mathrm{O}(e^2). \tag{4.6}$$

We chose $e\mathscr{F}$ as the scaling parameter in (4.2). Why not $e^\alpha\mathscr{F}$? We set $\alpha=1$ firstly because we remarked in Sec.II that e always multiplies $F_{\mu\nu}$ so that large amplitude variations of $F_{\mu\nu}$ can just as well be studied in the strong coupling limit; setting $\alpha\neq 1$ breaks this correspondence. Secondly, if $\alpha>1$ then the lower bound in (4.6) would be more negative, hence not optimal. If $\alpha<1$ one gets a better bound in (4.6) but the truncation error in (4.2) increases faster than e^2 for terms of $\mathrm{O}(F^4)$ and higher order. So $\alpha=1$ is the unique choice. The scaling parameter is further discussed in Sec.VI A.

The lower bound in (4.6) is related to and in argeement with the constant magnetic field growth of scalar QED's effective action [17]

$$S_{\mathrm{SQED}}=-\ln\det{}_{\mathrm{SQED}}=\frac{B^2V}{96\pi^2}e^2\ln\left(\frac{eB}{m^2}\right)+\mathrm{O}(e^2). \tag{4.7}$$

where V is a four-dimensional volume cutoff.

This completes the discussion of the growth of the first term in (2.6) and (3.9). We now turn to the all-important second term.

V. STRONG COUPLING BEHAVIOR OF $\det_3$

A. Paramagnetic property of $\det_3$

In Appendix A it is shown that $\Delta_A^{1/2}\sigma F\Delta_A^{1/2}\equiv T$ belongs to the trace ideal $\mathscr{I}_3$ for $F_{\mu\nu}\in\cap_{p>2}L^p(\mathbb{R}^4)$ and $m>0$. This means that T is a compact operator that, in our case, maps $L^2(\mathbb{R}^4)$ into itself. Being compact its eigenvalues, $\{\lambda_n\}_{n=1}^\infty$, are discrete, and each has finite multiplicity. We order the λ_n by $|\lambda_1|\geq|\lambda_2|\geq\ldots>0$. Because $T\in\mathscr{I}_3$ the eigenvalues $\lambda_n\to 0$ and satisfy

$$\sum_{n=1}^{\infty}|\lambda_n|^3<\infty. \tag{5.1}$$

Finally, $\ln\det_3(1+T)$ is gauge invariant (Appendix D) and satisfies by (3.7)

$$\begin{aligned}&\ln\det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right)\\&\quad=\ln\det\left[(1+T)\exp\left(-T+\frac{1}{2}T^2\right)\right]\\&\quad=\mathrm{Tr}\left[\ln(1+T)-T+\frac{1}{2}T^2\right]\\&\quad=\sum_{n=1}^{\infty}\left[\ln(1+\lambda_n)-\lambda_n+\frac{1}{2}\lambda_n^2\right].\end{aligned} \tag{5.2}$$

In Appendix D it is shown that for every eigenstate of T with eigenvalue λ_n there is another with eigenvalue $-\lambda_n$. Therefore, (5.2) becomes

$$\begin{aligned}&\ln\det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right)\\&\quad=\sum_{n=1}^{\infty}\left[\ln(1-\lambda_n^2)+\lambda_n^2\right],\end{aligned} \tag{5.3}$$

where the sum is over positive eigenvalues. We will see in Sec.VII B that the condition on $F_{\mu\nu}$ can be relaxed somewhat.

Since $\ln\det_3$ is real and finite then $\lambda_n<1$ for all n. Hence,

$$\ln\det{}_3\left(1+\Delta_A^{1/2}\frac{1}{2}e\sigma F\Delta_A^{1/2}\right)\leq 0, \tag{5.4}$$

since $\ln(1-x^2)+x^2\leq 0$ for $0\leq x<1$. This inequality has a physical origin. Referring to (3.5) and (3.6) and simplifying exactly as outlined in Appendix C for the function Π we obtain

$$\begin{aligned}\ln\det{}_3=&\int_0^\infty\frac{\mathrm{d}t}{t}e^{-tm^2}\mathrm{Tr}\Big[e^{-tD^2}-e^{-t\left(D^2+\frac{1}{2}e\sigma F\right)}\\&+\frac{e^2}{8}te^{-tD^2/2}\sigma F\Delta_A^{1/2}\Delta_A^{1/2}\sigma Fe^{-tD^2/2}\Big].\end{aligned} \tag{5.5}$$

That $\ln\det_3<0$ is now seen as a consequence of the paramagnetism of a charged spin-1/2 fermion in a static, four-dimensional magnetic field $F_{\mu\nu}$: on average its energy levels are lowered by $F_{\mu\nu}$. This is made more precise by a version of the Peierls-Bogoliubov inequality derived from Klein's inequality [18–20]:

$$\mathrm{Tr}\Big(e^{-t(P-eA)^2}-e^{-\left[(P-eA)^2+\frac{1}{2}e\sigma F\right]t}\Big)\leq 0. \tag{5.6}$$

The last term in (5.5) has been purposely written in the form $U^\dagger U$ and is therefore positive. Nevertheless, it is dominated by the paramagnetism of charged fermions through (5.6) which drives the integral in (5.5) to a negative value. This explains the second term in (2.6).

B. Lower bound on $\ln\det_3$ in the absence of zero modes

The eigenvalues in (5.3) are obtained from

$$\frac{e}{2}\Delta_A^{1/2}\sigma F\Delta_A^{1/2}\varphi_n = -\lambda_n\varphi_n, \tag{5.7}$$

where $\varphi_n \in L^2$. Let $\Delta_A^{1/2}\varphi_n = \psi_n$ and obtain

$$\left[(P-eA)^2 + \frac{e}{2\lambda_n}\sigma F\right]\psi_n = -m^2\psi_n, \tag{5.8}$$

where $\psi_n \in L^2$ as shown at the end of Appendix A. Eq. (5.8) illustrates the role of the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ as coupling constants whose discrete values result in bound states with energy $-m^2$ for a fixed value of e.

Because γ_5 commutes with σ, an eigenstate ψ_n of (5.8) has definite chirality. In the representation (D7) γ_5 is diagonal with elements $\pm\mathbb{1}_2$, and so we need only deal with the two-dimensional chirality eigenstates $\psi_n^\pm$.

We note that each eigenvalue $\lambda_n(e)$ is a bounded function of e as required by $|\lambda_n(e)| < 1$ for all finite values of of e. This is illustrated by the constant field case:

$$|\lambda_n| = \frac{|eB|}{(2n+1)|eB|+m^2}, \quad n = 0, 1, \ldots \tag{5.9}$$

Therefore, the series in (5.3) will tend to an e-independent limit for $e \gg 1$ unless the degeneracy of the eigenvalues increases with e. The special case of a zero mode supporting background potential that allows $|\lambda_n|$ to approach 1 arbitrarily closely for $e \gg 1$ will be considered in the next section.

To bound $\ln\det_3$ for $e \gg 1$ we will first estimate the eigenvalue degeneracy for the most symmetric case of an $\mathrm{O}(2)\times\mathrm{O}(3)$ background field. This estimate will place an upper bound on the eigenvalue degeneracy of any random field. The $\mathrm{O}(2)\times\mathrm{O}(3)$ symmetric fields have the standard form [21–23]

$$A_\mu(x) = M_{\mu\nu}x_\nu a(r), \tag{5.10}$$

where $M_{\mu\nu}$ is the antiself-dual antisymmetric matrix with nonvanishing elements $M_{12} = M_{30} = 1$ and $r^2 = x_\mu^2$. Alternatively M may be replaced with the self-dual antisymmetric matrix N with nonvanishing elements $N_{03} = N_{12} = 1$.

Choosing the matrix M the eigenstates of (5.8) have the form [23]

$$\psi_n = r^{-2j-3/2}\begin{pmatrix} \mathscr{D}^j_{M-\frac12,m}(x)\rho_1(r) \\ \mathscr{D}^j_{M+\frac12,m}(x)\rho_2(r) \\ (j+m)^{\frac12} r\rho_3(r)\mathscr{D}^{j-\frac12}_{M,m-\frac12}(x) - (j-m+1)^{\frac12}(\rho_4(r)/r)\mathscr{D}^{j+\frac12}_{M,m-\frac12}(x) \\ (j-m)^{\frac12} r\rho_3(r)\mathscr{D}^{j-\frac12}_{M,m+\frac12}(x) + (j+m+1)^{\frac12}(\rho_4(r)/r)\mathscr{D}^{j+\frac12}_{M,m+\frac12}(x) \end{pmatrix}, \tag{5.11}$$

where $\mathscr{D}^j_{m_1m_2}(x)$ are the four-dimensional rotation matrices [23–25] normalized so that

$$\int \mathrm{d}\Omega_4\, \mathscr{D}^{j*}_{m_1m_2}(x)\mathscr{D}^{j'}_{m_3m_4}(x) = \delta_{jj'}\delta_{m_1m_3}\delta_{m_2m_4}\frac{2\pi^2r^{4j}}{2j+1}, \tag{5.12}$$

and where $2j = 0, 1, \ldots$; $-j \le m_i \le j$. This paper follows the conventions of [23, 24]; closely related ones appear in [25]. The index n has been omitted from ρ_i. Inserting the two positive chirality components of (5.11) into (5.8) results in the following equations for $\rho_{1,2}$ [24],

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(2j+1)^2-\frac14}{r^2} + (4M \mp 2)ea + e^2r^2a^2 \pm \frac{e}{\lambda_n^+}(4a + r\frac{\mathrm{d}a}{\mathrm{d}r})\right]\rho_{1,2} = -m^2\rho_{1,2}, \tag{5.13}$$

where the upper (lower) sign applies to ρ_1 (ρ_2), and λ_n^+ denotes a positive chirality eigenvalue. Since $(P-eA)^2 + \frac{e}{2}\sigma F \ge 0$ it is the λ_n^+-dependent terms in (5.13) that are

responsible for bound states at $-m^2$. There is a sequence of eigenvalues $1 > \lambda_1^+ \geq \lambda_2^+ \geq \ldots > 0$ dependent on e, j, M, m, and the parameters specifying A_μ that result in bound state solutions of (5.13). They are independent of the quantum number m in (5.11), resulting in a $(2j+1)$-fold degeneracy. Inspection of (5.13) indicates that in the positive chirality sector

$$\begin{aligned}\frac{1}{2}(\sigma F)^+ &= \left(4a + r\frac{\mathrm{d}a}{\mathrm{d}r}\right)\sigma_3 \\ &\equiv V(r)\sigma_3.\end{aligned} \tag{5.14}$$

In general the degeneracy of the level at $-m^2$ has contributions from both ρ_1 and ρ_2. Consider ρ_1. Assume that a and a' are bounded functions of r. Inclusion of zero modes requires $\lim_{r\to\infty} r^2 a = \nu$, where we may assume $\nu > 0$ as discussed in Sec.C below. Then $r^2V(r)$ is a bounded function of r and

$$\inf\left[r^2 V(r)\right] = -K_1 > -\infty. \tag{5.15}$$

The λ_n^+-independent terms on the left-hand side of (5.13) form a positive operator whose controlling parameter is j for fixed e. Thus a bound state at $-m^2$ can exist only if

$$(2j+1)^2 < \frac{e}{\lambda_n^+}K_1 + \frac{1}{4}. \tag{5.16}$$

This is a necessary condition but obviously not a sufficient one. The maximum allowed value of j for all finite values of m^2 and a fixed value of M is $J_1 < \left(\frac{eK_1}{4\lambda_n^+} + \frac{1}{16}\right)^{\frac{1}{2}} - \frac{1}{2}$. Hence, the maximum degeneracy μ_{1n}^+ of eigenvalue λ_n^+ associated with ρ_1 for $\frac{eK_1}{\lambda_n^+} \geq 1$ is

$$\mu_{1n}^+ = \sum_{j=0,\frac{1}{2},\ldots}^{J_1} (2j+1) < 2\left[\left(\frac{eK_1}{4\lambda_n^+}\right)^{\frac{1}{2}} + 1\right]^2. \tag{5.17}$$

For the other positive chirality state $\mathscr{D}^j_{M+\frac{1}{2},m}\rho_2/r^{2j+\frac{3}{2}}$ inspection of (5.13) indicates that the bound state at $-m^2$ acquires an additional maximal degeneracy μ_{2n}^+ satisfying the bound in (5.17) with K_1 replaced with $K_2 = \sup(r^2V(r)) < \infty$. It may happen that either ρ_1 or ρ_2 has no bound states at $-m^2$.

Is the dependence of μ_{1n}^+, μ_{2n}^+ on λ_n^+ reasonable? As $\lambda_n^+ \searrow 0$ the potential wells in $\pm\frac{e}{\lambda_n^+}V(r)$ deepen, increasing the probability that such wells can support a bound state at $-m^2$. As the wells deepen the centrifugal barrier term in (5.13) can increase, thereby allowing larger values of j and hence higher degeneracy, consistent with our result (5.17).

In the negative chirality sector

$$\frac{1}{2}(\sigma F)^- = \begin{pmatrix} -\mathscr{D}^1_{00} & \sqrt{2}\mathscr{D}^{1*}_{01} \\ \sqrt{2}\mathscr{D}^1_{01} & \mathscr{D}^1_{00}\end{pmatrix}\frac{1}{r}\frac{\mathrm{d}a}{\mathrm{d}r}, \tag{5.18}$$

where $\mathscr{D}^1_{00} = x_0^2 + x_3^2 - x_1^2 - x_2^2$ and $\mathscr{D}^1_{01} = -\sqrt{2}(x_0 + ix_3)(x_2 - ix_1)$. Insertion of (5.18) and the two negative chirality components of (5.11) in (5.8) results in coupled equations for ρ_3 and ρ_4:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{4j^2 - \frac{1}{4}}{r^2} + 4Mea + e^2r^2a^2\right)\rho_3 + \frac{e}{\lambda_n^-}ra'\left(\sqrt{1 - \frac{M^2}{(j+\frac{1}{2})^2}}\rho_4 + \frac{M}{j+\frac{1}{2}}\rho_3\right) = -m^2\rho_3 \tag{5.19}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{4(j+1)^2 - \frac{1}{4}}{r^2} + 4Mea + e^2r^2a^2\right)\rho_4 + \frac{e}{\lambda_n^-}ra'\left(\sqrt{1 - \frac{M^2}{(j+\frac{1}{2})^2}}\rho_3 - \frac{M}{j+\frac{1}{2}}\rho_4\right) = -m^2\rho_4. \tag{5.20}$$

These equations can be decoupled for large j by a unitary transformation U on ρ_3, ρ_4. Let $U\rho = \varphi$ with $U_{33} = U_{44} = (\frac{1+M}{(j+\frac{1}{2})})^{\frac{1}{2}}/\sqrt{2}$ and $U_{34} = -U_{43} = (\frac{1-M}{(j+\frac{1}{2})})^{\frac{1}{2}}/\sqrt{2}$ so that the coupled terms in (5.19),(5.20)proportional to e/λ_n^- are transformed to $(e/\lambda_n^-)ra'\sigma_3\varphi$. Comparing this with (5.13) the same analysis used in the positive chirality case applies here. Thus, following (5.17) the maximum degeneracies μ_{3n}^-, μ_{4n}^-associated with the bound states φ_3, φ_4 at $-m^2$ are bounded by eK/λ_n^-, where K is an e-independent constant. This assumes $e/\lambda_n^- >> 1$ corresponding to large j.

We emphasize that the estimated maximum degeneracies above are for one level at $-m^2$. They are not an estimate of the number of bound states at energy $\leq -m^2$ which is expected to vary as e^2 for $F_{\mu\nu} \in L^2$ by theorem 2.15 in [26].

We now have estimates for the maximum degeneracy of eigenvalues $\lambda_n^\pm$ obtained from (5.8) for the most symmetric admissible background field given by (5.10). The above results place an upper bound on the eigenvalue degeneracy μ_n of any admissible random field, namely for $e \gg 1$

$$\mu_n(e) < \frac{ec}{\lambda_n}, \tag{5.21}$$

where λ_n is one of the random field's eigenvalues obtained from (5.8), and c is e-independent. The $1/\lambda_n$ dependence of its right-hand side is important because it results in the convergent series $\sum_{n>N}^\infty \lambda_n^3$ in (5.23) below, whatever

the field may be.

Consider the series in (5.3) and divide it into $\sum_{n=1}^{N} + \sum_{n>N}^{\infty}$, where $\lambda_n^2 < \frac{1}{2}$ for $N > n$, N sufficiently large. Note in this case that

$$\frac{1}{2} \le \left| \frac{\ln(1-\lambda_n^2) + \lambda_n^2}{\lambda_n^4} \right| < 1. \tag{5.22}$$

Thus for any admissible random field, excluding those that support a zero mode, there follows from (5.3), (5.21), and (5.22)

$$\begin{aligned} &\left| \ln \det{}_3 \left(1 + \Delta_A^{\frac{1}{2}} \frac{1}{2} e\sigma F \Delta_A^{\frac{1}{2}} \right) \right| \\ &< \sum_{n=1}^{N} \left| \ln(1-\lambda_n^2) + \lambda_n^2 \right| + \sum_{n>N}^{\infty} \lambda_n^4 \\ &< \sum_{n=1}^{N} \left| \ln(1-\lambda_n^2) + \lambda_n^2 \right| + ec \sum_{\substack{n>N \\ \text{no degeneracy}}}^{\infty} \lambda_n^3, \end{aligned} \tag{5.23}$$

where the third line of (5.23) is valid when $e \gg 1$. In the absence of zero modes $\lim_{e\to\infty} \lambda_1 < 1$ unlike the zero mode case discussed in Sec.C below. By (5.1) the infinite series on the right converges. Moreover, the $e \to \infty$ limit of this series is finite. Thus, there is a number M such that for $n > M$, $\lambda_n(e) < C_n(e)/n^{1/3+\epsilon}$, $\epsilon > 0$ and C_n is a bounded function of n and e with $\lim_{e\to\infty} C_n(e) < \infty$. Otherwise $\lambda_n < 1$ for any n cannot be satisfied. Accordingly, the right-hand series in (5.23) is uniformly convergent in e by the Weierstrass M test, allowing its $e \to \infty$ limit to be taken term-by-term and establishing our claim. The remaining series, $\sum_{n=1}^{N} |\ln(1-\lambda_n^2) + \lambda_n^2|$, is obviously bounded by e following (5.21), excluding zero modes. Combining (5.3), (5.21), (5.22) and (5.23) gives in the absence of zero modes

$$0 \ge \lim_{e\to\infty} \ln \det{}_3 \left(1 + \Delta_A^{\frac{1}{2}} \frac{1}{2} e\sigma F \Delta_A^{\frac{1}{2}} \right) / e > -C, \tag{5.24}$$

where $C > 0$ is an e-independent constant depending on the specific background field. C must be a linear function of $F_{\mu\nu}$ to preserve the correlation $eF_{\mu\nu}$.

C. Zero modes

Consideration is now given to potentials supporting L^2 zero modes of the Dirac operator $\not{P} - e\not{A}$. It is these potentials that provide the mechanism governing the stability of QED and its non-perturbative quantization.

The relevance of zero modes to stability arises as follows. Suppose A_μ supports a zero mode, $\psi_{\text{zero},n}$, where n denotes the quantum numbers required to specify it. It is an L^2 solution of

$$\left[(P - eA)^2 + \frac{e}{2}\sigma F \right] \psi_{\text{zero},n} = 0, \tag{5.25}$$

obtained from (5.8) by setting $\lambda_n = 1$, $m = 0$. We continue to assume $\lambda_n > 0$ as discussed in Sec. V A. Then (5.25) requires $\langle \text{zero}, n | \, \sigma F \, | \text{zero}, n \rangle < 0$. Refer to (5.8) and replace λ_n with a general eigenvalue λ and denote the corresponding eigenstate by $\psi_{\lambda,n}$. Assume $\langle \lambda, n | \, \sigma F \, | \lambda, n \rangle < 0$. Then from (5.8) and (5.25) there follows

$$\frac{e}{2} \left(\frac{1}{\lambda} - 1 \right) \langle \text{zero}, n | \, \sigma F \, | \lambda, n \rangle = -m^2 \langle \text{zero}, n | \lambda, n \rangle. \tag{5.26}$$

There is no *a priori* reason why the two sides of (5.26) should vanish if the quantum numbers of the two states are the same. Based on our limited knowledge of four-dimensional Abelian zero modes [24] they have a distinctive structure, and so the nonvanishing of $\langle \text{zero}, n | \lambda, n \rangle$ distinguishes the eigenstate $\psi_{\lambda,n}$ –and its eigenvalue λ– from other eigenstates obtained from (5.8).

Divide (5.26) by e. For $e \gg 1$ conclude that λ has the form

$$\lambda = 1 - \delta(e, n, m, L, \dots), \tag{5.27}$$

where $0 < \delta < 1$ and that for fixed m, $\delta \searrow 0$ for $e \to \infty$. L is a parameter with the dimension of length introduced by A_μ that can combine with m to form a dimensionless δ. This result requires that the states $\psi_{\lambda,n}$ be in one-to-one correspondence with the zero modes $\psi_{\text{zero},n}$. The eigenvalue λ will be discussed for an analytically solvable case in Sec. 5 E.

Insertion of (5.27) in (5.3) gives

$$\begin{aligned} \ln \det{}_3 &= \sum_n \sigma_n \\ &\times \left[-\ln \left(\frac{1-\delta}{\delta} \right) + \ln\left[(1-\delta)(2-\delta) \right] + (1-\delta^2) \right] + \dots, \end{aligned} \tag{5.28}$$

where the remainder in (5.28) is the contribution from eigenvalues bounded away from 1 discussed in the previous section; σ_n is the degeneracy of state n.. The sum in (5.28) is over the quantum numbers specifying the zero modes of A_μ. Write (5.26) in the form

$$\frac{1-\delta}{\delta} = \frac{e}{2m^2} \left| \frac{\langle \text{zero}, n | \sigma F | \lambda, n \rangle}{\langle \text{zero}, n | \lambda, n \rangle} \right|, \tag{5.29}$$

where

$$\left| \frac{\langle \text{zero}, n | \sigma F | \lambda, n \rangle}{\langle \text{zero}, n | \lambda, n \rangle} \right| \le K \mathscr{F}. \tag{5.30}$$

Eq. (5.30) assumes $F_{\mu\nu}(x)$ is a bounded function in which case K is an e-independent constant; $\mathscr{F}$ is the amplitude of $F_{\mu\nu}$ corresponding to the scaling parameter introduced

in Sec.IV. Inserting (5.29) in (5.28) gives for $e \to \infty$

$$\ln \det_3 = -\sum_n \sigma_n \times \left[\ln\left(\frac{e\mathscr{F}}{m^2}\right) + \ln\left|\frac{\langle \text{zero}, n|\sigma F|\lambda, n\rangle/\mathscr{F}}{\langle \text{zero}, n|\lambda, n\rangle}\right| - 2\ln 2 - 1\right] + \mathrm{O(e)}. \tag{5.31}$$

The $O(e)$ term is the contribution from the eigenvalues bounded away from 1 discussed in the previous section. Since

$$\sum_n \sigma_n = \#\text{ zero modes supported by } A_\mu, \tag{5.32}$$

if the number of zero modes increases as e^2 or faster then the result (5.31) will override the bound in (5.24) and possibly drive $\ln\det_{\text{ren}}$ in (3.9) negative. Clearly, these considerations are highly relevant to QED's non-perturbative quantization.

D. Counting zero modes

Following (5.31) and (5.32) it is of exceptional interest to know the maximum number of zero modes a potential can support. To begin we focus on the most symmetric admissible potentials (5.10). It is assumed that zero mode potentials within the class (5.10) will produce the maximum number due to their high symmetry and hence large number of degenerate states $\psi_{\text{zero},n}$. As pointed out in the previous section, eigenstates $\psi_{\lambda,n}$ of (5.8) with eigenvalue λ given by (5.27) will be in one-to-one correspondence with the states $\psi_{\text{zero},n}$. We would then expect that zero mode supporting potentials with lesser symmetry will have their zero mode number bounded by this most symmetric result. It turns out that this reasoning is not completely correct and that potentials with lesser symmetry can compete with those in (5.10). This is a huge advantage for QED's stability. We will begin with the potentials (5.10) and then explain why this reasoning has to be modified.

The zero modes supported by the potentials in (5.10) have been discussed in [24]. We continue to assume that a and a' are bounded functions of r and in addition $\lim_{r\to\infty} r^2 a = \nu$, $\nu \neq 0$. That is, A_μ must have a $1/r$ falloff. This ensures that the global chiral anomaly $\mathcal{A}$ is nonvanishing:

$$\mathcal{A} = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu} = \pm\frac{\nu^2}{2}, \tag{5.33}$$

where ${}^\star FF = \partial_\alpha(\epsilon_{\alpha\beta\mu\nu}A_\beta F_{\mu\nu})$. The $+(-)$ sign in (5.33) results in the case of matrix M (N) defined under (5.10). Without loss of generality we will assume $\nu > 0$. The nonvanishing of $\mathcal{A}$ indicates that $F_{\mu\nu}$ is not square-integrable. We repeat here that it is sufficient to assume $F_{\mu\nu} \in \cap_{p>2} L^p$ to define $\det_3$, and therefore it can accommodate zero modes.

Choosing the matrix M in (5.10) it is found that only the positive chirality sector has normalizable zero modes [24]. This is a particular example of a vanishing theorem: all normalizable zero modes of $\not{D}^2$ have only one chirality. There is no such general theorem in QED_4, unlike the non-Abelian case [27, 28] and QED_2 [29]. Up to a normalization constant these are [24]

$$\psi_{\text{zero}}(x) = \mathscr{D}^j_{-j,m}(x)e^{-e\int_{r_0}^r \mathrm{d}r\, ra(r)}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \tag{5.34}$$

Here $\exp\left[-e\int_{r_0}^r \mathrm{d}r\, ra(r)\right] = \rho_2$ in (5.11) when $M = -j-1/2$ and in (5.13) when in addition $m^2 = 0$ and $\lambda_n^+ = 1$. Eq. (5.34) and the assumption $a(r) \underset{r\to\infty}{\sim} \nu/r^2$ indicate that ψ^+ is square-integrable provided $e\nu > 2j+2$. Following (5.32),

$$\#\text{ zero modes} = \sum_{j=0,\frac{1}{2},\ldots}^{j_{\max}} (2j+1) = \frac{1}{2}[e\nu]^2 - \frac{1}{2}[e\nu], \tag{5.35}$$

where $[x]$ is the greatest integer less than x. Using (5.33) for $e\nu \gg 1$,

$$\begin{aligned}\#\text{ zero modes} &= \frac{1}{2}(e\nu)^2 + \mathrm{O}(e\nu) \qquad (1)\\ &= \frac{e^2}{16\pi^2}\left|\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu}\right| + \mathrm{O}(e\nu).\end{aligned} \tag{5.36}$$

If the matrix M is replaced with N in (5.10) the zero modes shift to the negative chirality sector. Therefore, (5.36) includes this case.

Given another potential with lesser symmetry than $\mathrm{O}(2)\times\mathrm{O}(3)$ and having the same chiral anomaly we tentatively conclude that its zero mode number is bounded by the right-hand side of (5.36). This assumes that all of the potential's zero modes have one chirality only.

More information about the zero mode number of less symmetric potentials can be obtained from the index theorem for non-compact Euclidean space-time [30],

$$n_+ - n_- - \frac{1}{\pi}\sum_l\left[\delta_l^+(0) - \delta_l^-(0)\right] = -\frac{e^2}{16\pi^2}\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu}, \tag{5.37}$$

where $n_\pm$ is the number of positive/negative chirality L^2 zero modes; $\delta_l^\pm(0) \in (0,\pi]$ are the zero energy scattering phase shifts for the Hamiltonians $H_\pm = \frac{1}{2}(1\pm\gamma_5)\not{D}^2$, and l denotes the quantum numbers required to specify the phase shifts. The sum over phase shifts gives the fractional discrepancy between the index and the chiral

anomaly. Consequently the sum in (5.37) grows more slowly than e^2 for $e \gg 1$. Based on (5.37) if there were a general vanishing theorem for QED_4 then the $\mathrm{O}(2)\times\mathrm{O}(3)$ result in (5.36) would continue to hold for potentials with lesser symmetry. This perhaps counterintuitive conclusion that two potentials with the same chiral anomaly–one with maximal symmetry, the other with lesser symmetry– have the same number of zero modes is related to their common asymptotic behavior. Without a vanishing theorem (5.37) implies that the total number of zero modes may exceed the chiral anomaly. Summarizing,

$$\begin{aligned}&\#\text{ zero modes supported by } A_\mu\\ &\geq \frac{e^2}{16\pi^2}\left|\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu}\right| + \Delta,\end{aligned} \tag{5.38}$$

with the inequality applying in the absence of a vanishing theorem and $\Delta/e^2 \to O$ for $e \to \infty$.

Insertion of (5.38) in (5.31) gives with (5.32)

$$\begin{aligned}&\ln\det_3\\ &\leq -\frac{1}{16\pi^2}\left|\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu}\right| e^2\ln\left(\frac{e\mathscr{F}}{m^2}\right) + R,\end{aligned} \tag{5.39}$$

with $R/(e^2\ln e) \to 0$ for $e \to \infty$, in which case the bound in (5.24) is overridden. As noted in Sec.A the negative sign in (5.39) is a consequence of the paramagnetism of a charged spin-$\frac{1}{2}$ fermion in a static, four-dimensional magnetic field.

E. Eigenvalue λ

Because of the possible far-reaching implications of (5.39) for the non-perturbative quantization of QED and the electroweak model it is important to have an analytic calculation of the eigenvalue λ in (5.27) for a few special cases to show that the formalism outlined in Secs. C and D can be implemented.

We consider a class of maximally symmetric zero mode supporting potentials (5.10) with profile function

$$a(r) = \begin{cases} \frac{C}{R^2}\left(\frac{r}{R}\right)^{\epsilon-2} + \frac{(2-\epsilon)C-2\nu}{R^3}r + \frac{(\epsilon-3)C+3\nu}{R^2}, & r \leq R\\ \frac{\nu}{r^2}, & r > R\end{cases} \tag{5.40}$$

It is constructed so that a and a', and hence $F_{\mu\nu}$ are continuous at $r = R$. The parameter $\epsilon \geq 2$ to ensure that $F \in \cap_{p>2}L^p$. The constant C can be positive or negative, and we continue to assume $\nu > 0$.

As noted in Sec. D the L^2 zero modes of (5.25) reside in the positive chirality sector with $M = -j - \frac{1}{2}$ for the potentials (5.10). A L^2 solution of (5.8) originating from the zero mode (5.34) is

$$\psi_\lambda(x) = \mathscr{D}^j_{-jm}(x)\frac{f(r)}{r^{2j+\frac{3}{2}}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{5.41}$$

where $f \equiv \rho_2$ in (5.13) now satisfies

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\frac{1}{4}-(2j+1)^2}{r^2} + 4\left(j+\frac{1}{\lambda}\right)ea - e^2r^2a^2 + \frac{e}{\lambda}r\frac{\mathrm{d}a}{\mathrm{d}r}\right]f = m^2 f, \tag{5.42}$$

with eigenvalue λ given (5.27) when $e \gg 1$. For $r > R$ let $f = r^{\frac{1}{2}}g$ so that (5.42) becomes

$$g'' + \frac{1}{r}g' - \left(m^2 + \frac{(2j+1-e\nu)^2 + 2\left(1-\frac{1}{\lambda}\right)e\nu}{r^2}\right)g = 0, \tag{5.43}$$

whose decaying solution is the modified Bessel function $K_\alpha(mr)$ with

$$\alpha = \left[(2j+1-e\nu)^2 + 2\left(1-\frac{1}{\lambda}\right)e\nu\right]^{\frac{1}{2}}. \tag{5.44}$$

The eigenvalue λ is fixed by the boundary condition at $r = R$:

$$\frac{Rf'(R)}{f(R)} = \frac{1}{2} + \frac{RK'_\alpha(mR)}{K_\alpha(mR)}. \tag{5.45}$$

The left-hand side of (5.45) is calculated from the solution of (5.42) for $0 \leq r \leq R$.

The analysis simplifies by assuming $mR \ll 1$. Let $e\nu = N + \Delta$, $N = 2, 3, \ldots;\ 0 < \Delta < 1$, $j = 0, \frac{1}{2}, \ldots, j_{\max}$ with $j_{\max} = (N-2)/2$ since L^2 zero modes exist only for $e\nu > 2j+2$. It is known that $\det_{\mathrm{ren}}$ has a branch point in m beginning at $m = 0$ [24] which is evident by the presence of K_α in (5.45). This leads to the following small mass expansions for $j = 0, \frac{1}{2}, \ldots, j_{\max} - \frac{1}{2}$ and $\alpha_0 = e\nu - 2j - 1 > 2$,

$$f = Bf_0\left(1 + m^2f_2 + m^4f_4 + \mathrm{O}\left(m^{2\alpha_0} \text{ or } m^6\right)\right), \tag{5.46}$$

$$\lambda = 1 - m^2\delta_2 - m^4\delta_4 + \mathrm{O}\left(m^{2\alpha_0} \text{ or } m^6\right); \tag{5.47}$$

for $j = j_{\max}$, $1 < \alpha_0 < 2$,

$$f = Bf_0\left(1 + m^2 f_2 + m^{2\alpha_0} f_{2\alpha_0} + \mathrm{O}\left(m^4\right)\right) \tag{5.48}$$
$$\lambda = 1 - m^2\delta_2 - m^{2\alpha_0}\delta_{2\alpha_0} + \mathrm{O}\left(m^4\right); \tag{5.49}$$

where α_0 is the $m = 0$ term in the expansion of α in (5.44), and B is a normalization constant. The expansion of δ in (5.27), (5.47)and(5.49) in powers of m must begin at m^2 to be consistent with the boundary condition (5.45). For all cases there is a $\mathrm{O}(m^2)$ term in the expansions of f and λ. The case $e\nu = 3, 4, \ldots$ is commented on in Appendix E. Here f_0 is the solution of (5.42) when $m = 0$, $\lambda = 1$ and $0 \le r \le R$,

$$f_0 = r^{2j+\frac{3}{2}} e^{-e\int_0^r \mathrm{d}s\, sa(s)}, \tag{5.50}$$

With these expansions the two sides of (5.45) can be matched in powers of m to obtain λ. The calculation is outlined in Appendix E.

For $mR << 1$, $e\nu > 2j + 2$ and $e \gg 1$ the calculation in Appendix E gives, following (E11) and (E12),

$$\lambda = 1 - \frac{2m^2/e}{\|(\sigma F(r_0))^+\|_1}(1 + \mathrm{O}(1/e)) + \mathrm{O}\left(\frac{m^4R^4}{e^2}\right), \tag{5.51}$$

where $(\sigma F)^+$ is the positive chirality component of σF in (5.14) that is responsible for the existence of zero modes, and r_0 is the unique root in the interval $0 < r < R$ of

$$4j + 3 - 2er^2a(r) = 0. \tag{5.52}$$

Here $\|(\sigma F)^+\|_1$ is the spin trace norm of $(\sigma F)^+$ defined for an operator A by $\|A\|_1 = \mathrm{Tr}(A^\dagger A)^{1/2}$. Because $(\sigma F)^+$ obtained from (5.14) and (5.40) is a smooth function, λ is a slowly varying function of j since $\mathrm{d}r_0/\mathrm{d}j = \mathrm{O}(1/e)$ from (5.52). For this special case we can count zero modes following (5.35), (5.36) and rewrite (5.39) as an equality. To leading order in m^2/e, δ in (5.27) can be read off from (5.51). This fixes the argument of the logarithm in (5.28) precisely:

$$\ln\det{}_3 = -\sum_{j=0}^{j_{\max}}(2j+1)\left[\ln\left(\frac{e\|(\sigma F(r_0(j)))^+\|_1}{2m^2}\right) + \mathrm{O}(1)\right] + R_1, \tag{5.53}$$

where $j_{\max} = [e\nu]/2 - 1$ and $\lim_{e\to\infty} R_1/(e^2\ln e) = 0$. The remainder R_1 includes contributions to $\det_3$ from eigenvalues bounded away from 1 as discussed in Sec.B. Defining an average $F_{\mu\nu}, \mathscr{F}$, by

$$\sum_{j=0}^{j_{\max}}(2j+1)\ln\|(\sigma F(r_0(j)))^+\|_1 \Big/ \sum_{j=0}^{j_{\max}}(2j+1) \equiv \ln\mathscr{F} \tag{5.54}$$

obtain from (5.35) and (5.36) for $e \gg 1$

$$\ln\det{}_3 = -\frac{e^2}{16\pi^2}\left|\int \mathrm{d}^4x\, {}^\star F_{\mu\nu}F_{\mu\nu}\right|\left[\ln\left(\frac{e\mathscr{F}}{2m^2}\right) + \mathrm{O}(1)\right] + R_2, \tag{5.55}$$

where R_2 contains a $\mathrm{O}(e\nu\ln(e\mathscr{F}))$ term from the $\mathrm{O}(e\nu)$ residue in (5.36) and satisfies the same limit as R_1. The result (5.55) overrides the bound (5.24).

VI. CHARGE RENORMALIZATION TERM IN $\ln\det_{\mathrm{ren}}$

A. Scaling parameter

Consider the last contribution to $\ln\det_{\mathrm{ren}}$ in (2.6) and (3.9), here designated as

$$\Pi = e^2\int_0^\infty \mathrm{d}t e^{-tm^2}\left[\frac{\|F\|^2}{32\pi^2 t} - \frac{1}{2}\,\mathrm{Tr}\left(e^{-tD^2}F_{\mu\nu}\Delta_A F_{\mu\nu}\right)\right]. \tag{6.1}$$

It is not obvious what to call the right-hand side of (6.1), but since $e^2\|F\|^2/(32\pi^2 t)$ is part of the on-shell charge renormalization subtraction in $\ln\det_{\mathrm{ren}}$ it will be referred to as the charge renormalization term. As in Sec.IV break the integral in (6.1) into $\int_0^{1/e\mathscr{F}}$ and $\int_{1/e\mathscr{F}}^\infty$, where $\mathscr{F}$ fixes the scale of the amplitude of $F_{\mu\nu}$. Then $\Pi = I_1 + I_2 + I_3$, where

$$I_1 = \frac{e^2\|F\|^2}{32\pi^2}\int_{1/e\mathscr{F}}^\infty \frac{\mathrm{d}t}{t}e^{-tm^2}, \tag{6.2}$$

$$I_2 = \frac{e^2}{32}\int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \times\left[\frac{\|F\|^2}{\pi^2 t} - 16\,\mathrm{Tr}\left(e^{-tD^2}F_{\mu\nu}\Delta_A F_{\mu\nu}\right)\right], \tag{6.3}$$

$$I_3 = -\frac{e^2}{2}\int_{1/e\mathscr{F}}^\infty \mathrm{d}t e^{-tm^2}\,\mathrm{Tr}\left(e^{-tD^2}F_{\mu\nu}\Delta_A F_{\mu\nu}\right). \tag{6.4}$$

At this point the choice of scaling parameter in (6.2)-(6.4) appears arbitrary. It is not for the following reasons.

(a) As remarked in Sec. IV, if the strong coupling behavior of $\det_{\mathrm{ren}}$ is to have anything to do with large amplitude variations of $F_{\mu\nu}$ then e must appear in the combination $e\mathscr{F}$.

(b) The scaling parameter must be universal and not tied to any particular background field. As m is always present in $\det_{\mathrm{ren}}$ it should be considered in the construction of a possible scaling parameter.

(c) The scaling parameter should result in the largest possible lower bound on Π for $e\mathscr{F} \gg m^2$.

(d) The lower bound should respect what is known about $\ln\det_{\rm ren}$'s mass dependence.

Based on (a)-(c) and the requirement that the scaling parameter have dimension (length)$^{-2}$ then possible scaling parameters have the form $(e\mathscr{F})^a m^b$, $2a + b = 2$, $a \neq 0$. But only $a = 1$, $b = 0$ are allowed by requirement (d). To see why consider I_1 in (6.2). Following the result (4.5) for $e\mathscr{F} \gg m^2$,

$$I_1 = \frac{e^2\|F\|^2}{32\pi^2} \ln\left(\frac{e\mathscr{F}}{m^2}\right) - \gamma + R, \tag{6.5}$$

where again γ is Euler's constant and $0 < |R| < m^2/(e\mathscr{F})$. The mass singularity in (6.5) is induced by the on-shell charge renormalization of $\ln\det_{\rm ren}$ in (2.1), the starting point of this analysis. It is shown in Appendix F that for potentials $A_\mu \in \bigcap_{r\geq 4-\epsilon} L^r(\mathbb{R}^4)$, $\epsilon > 0$ and arbitrarily small, $\ln\det_{\rm ren}$ at $m = 0$ is finite when it is renormalized off-shell. Moreover, its $m = 0$ limit is continuous. The restriction on A_μ excludes zero modes. Including them would cause $\ln\det_3$ to diverge at $m = 0$ as found in the results (5.31) and (5.39) that are independent of how $\ln\det_{\rm ren}$ is renormalized.

To define $\det_5$ in (F1), and therefore $\det_{\rm ren}$, it is sufficient to assume $A_\mu \in \bigcap_{r>4} L^r(\mathbb{R}^4)$ [7, 31]. The charge renormalization term Π depends only on D^2 and is therefore insensitive to zero modes. Without loss of generality we may assume here that $F_{\mu\nu} \in L^2$ and therefore that $A_\mu \in L^4$. This follows from the Sobolev inequality for gradients on $\mathbb{R}^4$ [32]. Hence the restriction on A_μ in the preceding paragraph can be consistently assumed here.

When the first term in (6.5) is combined with the mass singularity of $\ln\det_{SQED}$ in (4.6), multiplied by 2 as required by (3.9), obtain

$$\ln\det{}_{\rm ren} \underset{m\to 0}{\sim} -\frac{e^2\|F\|^2}{48\pi^2} \ln m^2 + \text{finite}. \tag{6.6}$$

The result in Appendix F allows us to state that this is the only divergent mass singularity of $\ln\det_{\rm ren}$ in the absence of zero modes. If $\ln\det_{\rm ren}$ were subtracted off-shell by adding to (2.1) the term

$$\frac{e^2\|F\|^2}{48\pi^2} \int_0^\infty \frac{\mathrm{d}t}{t}\left(e^{-t\mu^2} - e^{-tm^2}\right) = \frac{e^2\|F\|^2}{48\pi^2} \ln\left(\frac{m^2}{\mu^2}\right), \tag{6.7}$$

then $\ln\det_{\rm ren}$ would be finite at $m = 0$. This freedom to renormalize off-shell must be respected by the scaling parameter. Indeed, if the scaling parameter $(e\mathscr{F})^a m^b$, $b \neq 0$ were chosen in (4.2) and (6.2)-(6.4) then (6.6) would become

$$\ln\det{}_{\rm ren} \underset{m\to 0}{\sim} \left(\frac{b}{96} - \frac{1}{48}\right) \frac{e^2\|F\|^2}{\pi^2} \ln m^2 + \text{finite}. \tag{6.8}$$

This introduces a spurious $be^2\|F\|^2 \ln m^2/96\pi^2$ mass singularity into $\ln\det_{\rm ren}$'s lower bound when it is renormalized off-shell using (6.7). Therefore, the only acceptable scaling parameter for the strong coupling limit of Π in (6.1) and in $\det_{\rm SQED}$ in (4.2) is $e\mathscr{F}$. This further justifies the choice of scaling parameter in Sec.IV.

B. Estimates

Consider I_2 in (6.3). The trace in its last term can be put in the form $\mathrm{Tr}\,(A^\dagger A)$ using the trace's cyclic property. So the last term is not negative. Write out the trace term in its original form and note that

$$\begin{aligned}
&\int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \int \mathrm{d}^4x\mathrm{d}^4y e^{-tD^2}(x,y) F_{\mu\nu}(y)\Delta_A(y,x)F_{\mu\nu}(x) \\
&\leq \int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \int \mathrm{d}^4x \left|\left(e^{-tD^2} F_{\mu\nu}\Delta_A\right)(x)\right| |F_{\mu\nu}(x)| \\
&\leq \int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \int \mathrm{d}^4x \left(e^{-tP^2} |F_{\mu\nu}|\,|\Delta_A|\right)(x)\,|F_{\mu\nu}(x)| \\
&\leq \int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \int \mathrm{d}^4x \left(e^{-tP^2} |F_{\mu\nu}|\,\Delta\right)(x)\,|F_{\mu\nu}(x)| \\
&= \int_0^{1/e\mathscr{F}} \mathrm{d}t e^{-tm^2} \int \mathrm{d}^4x\mathrm{d}^4y e^{-tP^2}(x,y)\,|F_{\mu\nu}(x)|\,\Delta(y-x)\,|F_{\mu\nu}(x)|\,.
\end{aligned} \tag{6.9}$$

To obtain these results we used the diamagnetic inequality of Simon [12, 33] to go from the second to the third line:

$$\left|(e^{-tD^2} f)(x)\right| \leq \left(e^{-tP^2} |f|\right)(x). \tag{6.10}$$

This holds for all $t > 0$ and almost all $x \in \mathbb{R}^4$ and for potentials that are locally square integrable, as we are assuming. For more recent comments on (6.10) see [34]. In addition we used Kato's inequality in the form given by (A3) to go from the third to the fourth line.

Noting that

$$e^{-tP^2}(x,y) = \frac{1}{16\pi^2 t^2} e^{-|x-y|^2/4t}, \tag{6.11}$$

insertion of (6.9) in (6.3) gives

$$I_2 \geq \frac{e^2}{32\pi^2} \int_0^{1/e\mathscr{F}} \frac{\mathrm{d}t}{t} e^{-tm^2} \left(\|F\|^2 - \frac{1}{t} \int \mathrm{d}^4x \mathrm{d}^4y \, |F_{\mu\nu}(x)| \, \Delta(x-y) e^{-(x-y)^2/4t} \, |F_{\mu\nu}(y)| \right). \tag{6.12}$$

By Young's inequality in the form [19]

$$\left| \int \mathrm{d}^4x \mathrm{d}^4y f(x) g(x-y) h(y) \right| \leq \|f\|_p \|g\|_q \|h\|_r, \tag{6.13}$$

where $1/p + 1/q + 1/r = 2$, $p, q, r \geq 1$ and $\|f\|_p = (\int \mathrm{d}^4x \, |f(x)|^p)^{1/p}$, etc.,

$$I_2 \geq \frac{e^2 \|F\|^2}{32\pi^2} \int_0^{1/e\mathscr{F}} \frac{\mathrm{d}t}{t} e^{-tm^2} \left(1 - \frac{1}{t} \int \mathrm{d}^4x \Delta(x) e^{-x^2/4t} \right). \tag{6.14}$$

From $\Delta(x) = mK_1(mx)/(4\pi^2 x)$ and integral 2.16.8.5 of [35] get

$$I_2 \geq \frac{e^2 \|F\|^2}{32\pi^2} \int_0^{1/e\mathscr{F}} \frac{\mathrm{d}t}{t} e^{-tm^2} \left[1 - m^2 t e^{m^2 t} \Gamma(-1, m^2 t) \right], \tag{6.15}$$

where $\Gamma(-1, m^2 t)$ is the incomplete gamma function which we use in the form

$$\Gamma(-1, m^2 t) = \frac{1}{m^2 t} e^{-m^2 t} - \int_{m^2 t}^{\infty} \frac{\mathrm{d}z}{z} e^{-z}. \tag{6.16}$$

Insertion of (6.16) in (6.15) and integrating by parts gives for $e\mathscr{F} \gg m^2$

$$I_2 \geq \frac{e^2 \|F\|^2}{32\pi^2} \left[\frac{m^2}{e\mathscr{F}} \left(\ln\left(\frac{e\mathscr{F}}{m^2} \right) - \gamma + R \right) + 1 - e^{-m^2/e\mathscr{F}} \right], \tag{6.17}$$

with γ and R the same as in (6.5). Note that the lower bound in (6.17) is finite at $m = 0$ as it should be.

There are no ultraviolet divergences in I_2. The small t dependence of the first term in (6.3) is cancelled by the trace term, as was shown in the above non-perturbative estimate. So it must be a general property of the trace term that

$$16 \, \mathrm{Tr} \left(e^{-tD^2} F_{\mu\nu} \Delta_A F_{\mu\nu} \right) \underset{t \to 0}{\sim} \frac{\|F\|^2}{\pi^2 t} + \text{less singular in t.} \tag{6.18}$$

By inspection of (6.3) we conclude that

$$\lim_{e\mathscr{F} \to \infty} \frac{I_2}{(e\mathscr{F})^2 \ln(e\mathscr{F})} = 0. \tag{6.19}$$

Now consider I_3 in (6.4). As noted in the case of I_2 the trace is positive so that $I_3 \leq 0$. Application of the inequality (6.10) does not lead to a satisfactory lower bound on I_3 . Namely, if it were saturated I_3 would cancel the large amplitude growth of I_1 in (6.5), resulting in a slow $O((e\mathscr{F})^2)$ growth of Π in (6.1) and leading to the uninformative bound $\ln \det_{\mathrm{ren}} \geq -e^2 \|F\|^2 \ln(e\mathscr{F}/m^2)/96\pi^2 + O\left((e\mathscr{F})^2\right)$ following (3.9) and (4.6). We are confident that $\ln \det_{\mathrm{ren}}$ grows at least as fast as $ce^2 \|F\|^2 \ln(e\mathscr{F})$, $c > 0$, in the absence of zero mode supporting background fields. This confidence is based on the result [36] for the growth of $\ln \det_{\mathrm{ren}}$ for random, square-integrable, time-independent, non-zero mode supporting magnetic fields $\mathbf{B}(x)$ on $\mathbb{R}^3$,

$$\lim_{e \to \infty} \frac{\ln \det_{\mathrm{ren}}}{e^2 \ln e} = \frac{\|\mathbf{B}\|^2 T}{24\pi^2}, \tag{6.20}$$

where $\|\mathbf{B}\|^2 = \int \mathrm{d}^3x \mathbf{B} \cdot \mathbf{B}(x)$ and T is the size of the time box. Therefore, our estimate of I_3 has to be more detailed than in the case of I_2. We claim that $\lim_{e \to \infty} I_3/(e^2 \ln e) = 0$ for the class of fields considered here.

By summing over a complete set of scattering eigenstates $|E, \alpha\rangle$ of D^2, I_3 can be represented as

$$
\begin{aligned}
I_3 &= -\frac{e^2}{2}\sum_{\alpha,\beta}\int_{1/e\mathscr{F}}^{\infty}\mathrm{d}t e^{-tm^2}\int_0^{\infty}\mathrm{d}E e^{-tE}\int_0^{\infty}\mathrm{d}E'\frac{\langle E,\alpha|\,F_{\mu\nu}\,|E',\beta\rangle\,\langle E',\beta|\,F_{\mu\nu}\,|E,\alpha\rangle}{E'+m^2}\\
&= -\frac{e^2}{2}\sum_{\alpha,\beta}\int_0^{\infty}\mathrm{d}E\int_0^{\infty}\mathrm{d}E' e^{-(E+m^2)/e\mathscr{F}}\frac{|\langle E,\alpha|\,F_{\mu\nu}\,|E',\beta\rangle|^2}{(E+m^2)(E'+m^2)},
\end{aligned}
\tag{6.21}
$$

where α and β are complete sets of angular momentum-like quantum numbers. Due to the above theorem on the $m = 0$ limit of $\ln\det_{\mathrm{ren}}$ I_3 is finite at $m = 0$. So whether $F_{\mu\nu}$ is long or short-ranged is irrelevant to the growth of I_3 with e. Without loss of generality we may confine this discussion to fields with compact support. As $F_{\mu\nu}$ was assumed to be differentiable in previous sections the compactly supported fields are assumed to rapidly and smoothly tend to zero in a narrow zone near their boundries. In addition we may assume rotational symmetry. Asymmetric, tangled fields will tend to lower the matrix elements $|\langle E,\alpha|\,F_{\mu\nu}\,|E',\beta\rangle|$. We will assume maximally symmetric $O(3)XO(2)$ fields to maximize $|I_3|$.

For the potentials (5.10) the equation for the radial part of the scattering states that satisfy $D^2\psi_{E,\alpha} = E\psi_{E,\alpha}$ is [24]

$$
\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{(2j+1)^2-1/4}{r^2}+4m_1ea+e^2r^2a^2\right)\phi_{Ejm_1}(r) = E\phi_{Ejm_1}(r),
\tag{6.22}
$$

where $\psi_{E,\alpha}(x) = r^{-2j-3/2}\phi_{Ejm_1}(r)\mathscr{D}^j_{m_1m_2}(x), r = |x|$, and the four-dimensional rotation matrices $\mathscr{D}^j_{m_1m_2}$ are defined in Sec. V.B. Let $F_{\mu\nu}$ have range R. For $r > R$ the normalized wave function is, on setting the chiral anomaly equal to zero in [24],

$$
\begin{aligned}
\phi_{Ejm_1}(r) = \sqrt{\frac{r}{2}}J_{2j+1}(kr)\cos\delta_{jm_1}(k,e)\\
-\sqrt{\frac{r}{2}}Y_{2j+1}(kr)\sin\delta_{jm_1}(k,e),
\end{aligned}
\tag{6.23}
$$

where $\delta_{jm_1}(k,e)$ is the scattering phase shift in the indicated channel, $E = k^2$, and Y_n is a Bessel function of the second kind.

We assumed in Sec. V.B that a and ra' are bounded functions of r. This will be assumed here. Therefore, any admissible a maintains the small distance behavior $\phi_{Ejm_1}\sim r^{2j+3/2}$ independent of e. What ϕ_{Ejm_1} does as $r\nearrow R$ is manifested in the exterior wave function (6.23) through the phase shifts. From (6.22), although a descends rapidly to zero in a zone near $r = R$, it is evident from the $(era)^2$ term in (6.22) that as $e\to\infty$ there develops a high barrier at some point $r < R$ that blocks the entry of the exterior wave function (6.23), resulting in approximate hard sphere scattering. This happens however rapidly $F_{\mu\nu}$ varies for $r < R$. So there is no reason why $F_{\mu\nu} =$ constant for $r < R$ and falling rapidly to zero just before $r = R$ cannot be taken as representative of the general field case for the strong coupling estimate of I_3.

Accepting this, refer to (5.10) and set $a(r) = \lambda/R^2$ for $0\le r\le R-\epsilon$ and $a(R) = 0$. Then $F_{\mu\nu} = 2\lambda M_{\mu\nu}/R^2$ for $0 < r < R-\epsilon$. The parameter λ is related to the scaling parameter $\mathscr{F}$ by $\mathscr{F}^2 = F^2_{\mu\nu} = 16\lambda^2/R^4$ since $M^2_{\mu\nu} = 4$. Then

$$
\begin{aligned}
&\langle Ejm_1|\,F_{\mu\nu}\,|E'j'm'_1\rangle\\
&= \frac{4\pi^2\lambda M_{\mu\nu}}{2j+1}\delta_{jj'}\delta_{m_1m'_1}\int_0^R \mathrm{d}r\phi_{Ejm_1}\phi_{E'jm_1},
\end{aligned}
\tag{6.24}
$$

where we have taken the limit $\epsilon = 0$ on the right-hand side of (6.24). As shown below it follows from (6.22) that

$$
\begin{aligned}
&\left(\phi_{E'jm_1}\phi'_{Ejm_1}-\phi_{Ejm_1}\phi'_{E'jm_1}\right)(R)\\
&= (E'-E)\int_0^R \mathrm{d}r\phi_{Ejm_1}\phi_{E'jm_1}.
\end{aligned}
\tag{6.25}
$$

Then (6.24) and (6.25) combined with (6.21) give

$$
\begin{aligned}
I_3 =& -2\pi^4(e\mathscr{F})^2\int_0^\infty \mathrm{d}E\int_0^\infty \mathrm{d}E' e^{-(E+m^2)/e\mathscr{F}} \\
&\times \sum_{j=0,1/2,..}^{\infty}\frac{1}{(2j+1)^2}\sum_{m_1,m_2=-j}^{j}\frac{\left[\left(\phi_{E'jm_1}\phi'_{Ejm_1}-\phi_{Ejm_1}\phi'_{E'jm_1}\right)(R)\right]^2}{(E+m^2)(E'-E)^2(E'+m^2)}.
\end{aligned}
\tag{6.26}
$$

To obtain (6.25) from the assumed behavior of $F_{\mu\nu}$ multiply (6.22) at energy E by $\phi_{E'jm_1}(r)F_{\mu\nu}(r)$, subtract the result with $E \leftrightarrow E'$ and integrate by parts over the interval $0 \le r \le R$. Since $F_{\mu\nu}(R)=0$ and $\phi_{Ejm_1}(0)=0$ this gives

$$
\begin{aligned}
&\int_{R-\epsilon}^{R}\mathrm{d}r\left(\phi_{E'jm_1}\phi'_{Ejm_1}-\phi_{Ejm_1}\phi'_{E'jm_1}\right)\frac{\mathrm{d}F_{\mu\nu}(r)}{\mathrm{d}r} \\
&\quad=(E-E')\left[\frac{2\lambda M_{\mu\nu}}{R^2}\int_0^{R-\epsilon}\mathrm{d}r\phi_{Ejm_1}\phi_{E'jm_1}+\int_{R-\epsilon}^{R}\mathrm{d}r\phi_{Ejm_1}\phi_{E'jm_1}F_{\mu\nu}(r)\right].
\end{aligned}
\tag{6.27}
$$

Assuming $\epsilon/R \ll 1$ and noting that $\int_{R-\epsilon}^{R}\mathrm{d}rF'_{\mu\nu}(r) = -F_{\mu\nu}(R-\epsilon) = -\frac{2\lambda M_{\mu\nu}}{R^2}$, (6.25) follows after letting $\epsilon \to 0$.

The phase shifts required to calculate I_3 are obtained as follows. Set $a = \lambda/R^2$ in (6.22) and let, omitting subscripts,

$$
\phi(r) = r^{2j+3/2}f(r)e^{-\lambda e r^2/2R^2}. \tag{6.28}
$$

Then

$$
f''+\left(\frac{4j+3}{r}-\frac{2\lambda e r}{R^2}\right)f'+\left[k^2-\frac{4\lambda e}{R^2}(j+m_1+1)\right]f = 0. \tag{6.29}
$$

The solution of (6.29) regular at the origin is the confluent hypergeometric function

$$
f(r) = M\left(j+m_1+1-\frac{(kR)^2}{4\lambda e}, 2j+2, \frac{\lambda e r^2}{R^2}\right), \tag{6.30}
$$

following the notation of [37]. Joining (6.23) with (6.28) at $r=R$ gives

$$
\tan\delta_{jm_1}(k,\lambda\mathrm{e}) = \frac{(\gamma-1/2)J_{2j+1}(kR)-kRJ'_{2j+1}(kR)}{(\gamma-1/2)Y_{2j+1}(kR)-kRY'_{2j+1}(kR)}, \tag{6.31}
$$

where $\gamma = (r\phi'/\phi)_R$. Eqs. (6.28), (6.30) and Eq.(13.4.8) in [37] for $\mathrm{d}M(a,b,z)/\mathrm{d}z$ give

$$
\gamma = 2j+\frac{3}{2}-\lambda e+\frac{2\lambda e a}{b}\frac{M(a+1,b+1,\lambda e)}{M(a,b,\lambda e)}, \tag{6.32}
$$

where $a = j+m_1+1-(kR)^2/(4\lambda e), b = 2j+2$. There are several cases. For $j < \lambda e \gg 1$, fixed k,

$$
\begin{aligned}
\gamma =& \lambda e+2m_1-\frac{1}{2}-\frac{(kR)^2}{2e\lambda} \\
&+O\left(\frac{j^2}{\lambda e},\frac{j(kR)^2}{(\lambda e)^2},\frac{(kR)^4}{(\lambda e)^3}\right).
\end{aligned}
\tag{6.33}
$$

For $j > \lambda e \gg 1$, fixed k,

$$
\gamma = 2j+\frac{m_1}{j}\lambda e-\frac{(kR)^2}{4j}+O\left(\frac{\lambda e}{j^2},\frac{m_1\lambda e}{j^2},\frac{(kR)^2}{j^2}\right), \tag{6.34}
$$

and for $kR \to \infty$, fixed j, λe,

$$
\gamma = -\left[kR+O\left(\frac{1}{kR}\right)\right]\frac{J_{2j+2}(kR)}{J_{2j+1}(kR)}+2j-\lambda e+3/2. \tag{6.35}
$$

These results are obtained using the asymptotic expansions of $M(a,b,z)$ for large a,b,z in [37, 38]. Following (6.35) the phase shifts vanish at high energy as $\tan\delta \sim (e\lambda/kR)\cos^2(kR-\left(j+\frac{1}{2}\right)\pi-\pi/4)$.

In order to estimate I_3 for $e\mathscr{F} \to \infty$ it is convenient to divide the range of the kR, $k'R$ integrations in (6.26) into $[0,2)$, $[2,2\sqrt{e\mathscr{F}R^2})$, $[2\sqrt{e\mathscr{F}R^2}, 2(e\mathscr{F}R^2)^{1-\epsilon})$, $[2e\mathscr{F}R^2,\infty]$ and the special case $kR, k'R = O(e\mathscr{F}R^2)^{1-\epsilon}$, where $0<\epsilon \ll 1$. To accommodate the joining conditions (6.33)-(6.35) the range of j also has to be partitioned. It is essential not to interchange the large $e\mathscr{F}$ limit with the sum over j. We find that the dominant contributions to (6.26) come from $0 \le j \le \sqrt{e\mathscr{F}R^2}, 2 \le kR \lesssim O(\sqrt{e\mathscr{F}R^2})$ and

$2 \leq k'R \leq \infty$. There are many cases to consider; we outline here a representative case to indicate how the estimates are done.

Consider the contribution to (6.26) given by

$$I \equiv -8\pi^4(e\mathscr{F}R^2)^2 \sum_{j=0}^{\sqrt{e\mathscr{F}R^2}} \sum_{m_1,m_2=-j}^{j} \frac{1}{(2j+1)^2} \int_{2j+1}^{2\sqrt{e\mathscr{F}R^2}} \frac{\mathrm{d}(kR)}{kR} e^{-\frac{k^2}{e\mathscr{F}}} \int_{2\sqrt{e\mathscr{F}R^2}}^{2(e\mathscr{F}R^2)^{1-\epsilon}} \frac{\mathrm{d}(k'R)}{k'R} \times \frac{\left[\left(\phi_{E'jm_1}\phi'_{Ejm_1} - \phi_{Ejm_1}\phi'_{E'jm_1}\right)(R)\right]^2}{R^4(k'^2-k^2)^2}, \tag{6.36}$$

where we have noted above that we can set $m = 0$. For the range of kR, $k'R$ and j in (6.36) joining condition (6.33) applies. From (6.23), (6.31) and (6.33) obtain

$$\frac{(\phi_{E'}\phi'_E - \phi_E\phi'_{E'})(R)}{k'^2-k^2} \underset{\lambda e \gg 1}{\sim} \frac{R^2}{2\pi^2(\lambda e)^3}\left[\left(J_{2j+1}(kR) - \frac{kR}{\gamma-1/2}J'_{2j+1}(kR)\right)^2 + \left(Y_{2j+1}(kR) - \frac{kR}{\gamma-1/2}Y'_{2j+1}(kR)\right)^2\right]^{-1/2} \times\left[k \to k'\right]^{-1/2} \sim \frac{R^2}{2\pi^2(\lambda e)^3}\left(J^2_{2j+1}(kR) + Y^2_{2j+1}(kR)\right)^{-1/2}\left(J^2_{2j+1}(k'R) + Y^2_{2j+1}(k'R)\right)^{-1/2}. \tag{6.37}$$

Hence,

$$I \underset{e\lambda \gg 1}{\sim} -\frac{8192}{(e\mathscr{F}R^2)^4} \sum_{j=0}^{\sqrt{e\mathscr{F}R^2}} \int_{2j+1}^{2\sqrt{e\mathscr{F}R^2}} \frac{\mathrm{d}(kR)}{kR} e^{-k^2/e\mathscr{F}} \frac{1}{J^2_{2j+1}(kR) + Y^2_{2j+1}(kR)} \times \int_{2\sqrt{e\mathscr{F}R^2}}^{2(2\mathscr{F}R^2)^{1-\epsilon}} \frac{\mathrm{d}(k'R)}{k'R} \frac{1}{J^2_{2j+1}(k'R) + Y^2_{2j+1}(k'R)}, \tag{6.38}$$

where the sums over m_1 and m_2 have been taken. To estimate (6.38) use Watson's inequality (Eq.(1), Sec.13.74 of [39])

$$\frac{2}{\pi x} < J^2_n(x) + Y^2_n(x) < \frac{2}{\pi}(x^2-n^2)^{-1/2}, \tag{6.39}$$

for $x \geq n \geq 1/2$. This is used repeatedly in our estimates. An easy calculation gives

$$I \underset{e\lambda \gg 1}{=} O\left(-(e\mathscr{F}R^2)^{-2-\epsilon}\right), \tag{6.40}$$

with $0 < \epsilon \ll 1$. The remaining contributions to I_3 give

$$I_3 \underset{e\lambda \gg 1}{=} O\left(-(e\mathscr{F}R^2)^{-2}\right), \tag{6.41}$$

or smaller as in (6.40). The dominant estimate in (6.41) comes from the intervals $0 \leq j \leq \sqrt{e\mathscr{F}R^2}, 2j+1 \leq kR \leq O((\sqrt{e\mathscr{F}R^2})$, $O(e\mathscr{F}R^2) \leq k'R \leq \infty$.

We have given reasons above why this calculation of the large amplitude growth of I_3 is representative. In view of (6.41) we are confident that

$$\lim_{e\mathscr{F}\to\infty} \frac{I_3}{(e\mathscr{F})^2\ln(e\mathscr{F})} = 0, \tag{6.42}$$

for all admissible random fields. Combining (6.1), (6.5), (6.19) and (6.42) we obtain for large amplitude variations of admissible random fields $F_{\mu\nu}$

$$\Pi \underset{e\mathscr{F}\to\infty}{=} \frac{e^2\|F\|^2}{32\pi^2}\ln\left(\frac{e\mathscr{F}}{m^2}\right) + R_1, \tag{6.43}$$

with $\lim_{e\mathscr{F}\to\infty} R_1/[(e\mathscr{F})^2 \ln(e\mathscr{F})] = 0$. The term "admissible random field" is discussed in Sec. VII.

C. Summary

In the absence of zero mode supporting random background fields (3.9), (4.6), (5.24) and (6.43) give the final result

$$\ln\det_{ren} \underset{e\mathscr{F}\to\infty}{\geq} \frac{1}{48\pi^2} e^2 \|F\|^2 \ln\left(e\mathscr{F}/m^2\right) + R_2, \quad (6.44)$$

with R_2's growth bounded as R_1's above. The $\ln m^2$ contribution to (6.44) is due to on-shell charge renormalization. For off-shell renormalization m^2 is replaced with a subtaction parameter μ^2 as discussed in Sec. A above.

If zero mode supporting background fields are included and all of the zero modes have the same chirality then by (3.9), (4.6), (5.39)(an equality in this case) and (6.43).

$$\begin{aligned}\ln\det{}_{\mathrm{ren}} \underset{e\mathscr{F}\to\infty}{\geq} & -\frac{1}{16\pi^2}\left|\int \mathrm{d}^4x {}^*F_{\mu\nu}F_{\mu\nu}\right| e^2 \\ & \times \ln(\frac{e\mathscr{F}}{m^2}) + \frac{1}{48\pi^2}e^2\|F\|^2 \ln(\frac{e\mathscr{F}}{m^2}) + R_3,\end{aligned} \quad (6.45)$$

with R_3 bounded as R_1 and R_2 above. Recall that $\int \mathrm{d}^4x {}^*F_{\mu\nu}F_{\mu\nu}/16\pi^2$ is the chiral anomaly.

If the zero modes supported by A_μ have both positive and negative chirality there is no counting theorem and (6.45) is replaced with, following (5.31) and (5.32),

$$\begin{aligned}\ln\det{}_{\mathrm{ren}} \underset{e\mathscr{F}\to\infty}{\geq} & -(\#\text{zero modes supported by } A_\mu) \\ & \times \ln(\frac{e\mathscr{F}}{m^2}) + \frac{1}{48\pi^2}e^2\|F\|^2 \ln(\frac{e\mathscr{F}}{m^2}) + R_4.\end{aligned} \quad (6.46)$$

The number of zero modes grows at least as fast as e^2 following (5.37), provided the chiral anomaly is non-zero.. If they grow as e^2 or less then $\lim_{e\mathscr{F}\to\infty} R_4/[(e\mathscr{F})^2\ln(e\mathscr{F})] = 0$.

Known 4D Abelian zero modes require $F_{\mu\nu} \notin L^2$. So the $\|F\|^2$ terms in (6.45) and (6.46) need a volume cutoff that will be discussed in Sec.VII. Assuming in this section that $F_{\mu\nu} \in L^2$ served its purpose to obtain the structure of the charge renormalization term's large field amplitude contribution to $\ln\det_{\mathrm{ren}}$.

An assumption underlying (6.46) is that all admissible 4D Abelian zero mode supporting potentials have a $1/|x|$ falloff as $|x| \to \infty$. If there were zero mode supporting potentials whose falloff is faster than $1/|x|$ the associated chiral anomaly would vanish since ${}^*F_{\mu\nu}F_{\mu\nu} = \partial_\alpha(\epsilon_{\alpha\beta\mu\nu}\mathrm{A}_\beta F_{\mu\nu})$. The vanishing of the right-hand side of (5.37) implies $n_+ = n_-$. Without being able to place a lower bound on the number of zero modes (6.46) loses its predictive power in this case. A 4D Abelian vanishing theorem stating that all normalizable zero modes have either positive or negative chirality, as in QCD_4, needs to be either proved or falsified by a counterexample.

Further discussion of (6.44)-(6.46) appears at the end of Sec. VII.

VII. REGULARIZATION

In principle $\det_{ren}$ can be calculated as an explicit function of $F_{\mu\nu}$ before inserting it into the functional integral (2.5). The input potentials must correspond to random potentials supported by $\mathrm{d}\mu_0(A)$. It is generally accepted that these belong to $\mathscr{S}'(\mathbb{R}^4)$, the space of tempered distributions. This is the first requirement.

Throughout we have assumed smooth potentials, including zero mode supporting potentials $A_\mu(x)$ with a $1/|x|$ falloff for $|x| \to \infty$. In Sec.VA it was assumed that $F_{\mu\nu} \in \bigcap_{r>2} L^r(\mathbb{R}^4)$ which we noted may be too strong a condition. The $L^p(\mathbb{R}^4)$ Sobolev inequality $\|\nabla f\|_p \geq K\|f\|_q$, where K is a constant and $q = 4p/(4-p), 1 < p < 4$ [32], implies $A_\mu \in \bigcap_{r>4} L^r(\mathbb{R}^4)$ when A_μ is once differentiable and $F_{\mu\nu} \in \bigcap_{>2}^{<4} L^r(\mathbb{R}^4)$. This condition on A_μ and the weaker condition on $F_{\mu\nu}$ are sufficient to define $\det_5$ in (F1) to ensure that $\ln\det_{\mathrm{ren}}$ is defined when $m \neq 0$ [7, 31]. These assumptions constitute the second requirement.

The final requirement is that an ultraviolet cutoff mechanism be introduced.

These three requirements can be satisfied by calculating $\ln\det_{\mathrm{ren}}$ in terms of the potentials

$$A_\mu^\Lambda(x) = \int \mathrm{d}^4y f_\Lambda(x-y)A_\mu(y), \quad (7.1)$$

where $A_\mu \in \mathscr{S}'(\mathbb{R}^4)$ and $f_\Lambda \in \mathscr{S}(\mathbb{R}^4)$, the space of functions of rapid decrease. Then $A_\mu^\Lambda \in \mathrm{C}^\infty$. Besides smoothing A_μ, (7.1) also introduces a sequence of ultraviolet cutoffs. Thus, from (2.3) conclude that

$$\int \mathrm{d}\mu_0(A)A_\mu^\Lambda(x)A_\nu^\Lambda(y) = D_{\mu\nu}^\Lambda(x-y), \quad (7.2)$$

where the Fourier transform of the regularized free photon propagator in a fixed gauge is $\hat{D}_{\mu\nu}(k)|\hat{f}_\Lambda(k)|^2$ with $\hat{f}_\Lambda \in \mathrm{C}_0^\infty$, the space of C^∞ functions with compact support. For example, one might choose $\hat{f}_\Lambda = 1$, $k^2 \leq \Lambda^2$ and $\hat{f}_\Lambda = 0, k^2 \geq n\Lambda^2, n > 1$.

We note that if A_μ is a zero mode supporting potential then so is A_μ^Λ. Thus, if A_μ has a $1/|x|$ falloff then so does A_μ^Λ . This follows since the small-p dependence of their Fourier transforms, and hence their large-x dependence, are the same when $\hat{f}_\Lambda$ is chosen as above; chirality is preserved. Other mappings with the convolution in (7.1) can be followed with Young's inequality in the form (A7) with $s = 1$; the above conditions on A_μ and $F_{\mu\nu}$ are preserved.

Summarizing, we are instructed to replace all potentials and fields in this analysis with the smoothed potentials A_μ^Λ and fields $F_{\mu\nu}^\Lambda = \partial_\mu A_\nu^\Lambda - \partial_\nu A_\mu^\Lambda$, including the general representation (2.5). This allows the assumed restrictions on A_μ and $F_{\mu\nu}$ leading to (6.44)-(6.46) to be transferred to A_μ^Λ and $F_{\mu\nu}^\Lambda$ while keeping the underlying rough potentials A_μ in place.

The measure $\mathrm{d}\mu_0(A)$ is not modified. The substitution of A_μ^Λ for A_μ does not affect the analysis of Secs.V A-D. In particular, in Sec.V B where use is made of (5.10) we have

$$\begin{aligned}\hat{A}_\mu(k) &= M_{\mu\nu}\int \mathrm{d}^4x \mathrm{e}^{-ikx} x_\nu a(r)\\ &= iM_{\mu\nu}\partial_\nu \hat{a}(|k|).\end{aligned} \tag{7.3}$$

Then

$$\begin{aligned}A_\mu^\Lambda(x) &= \int \mathrm{d}^4y f_\Lambda(x-y)A_\mu(y)\\ &= (a_\Lambda(r) + \mathrm{h}_\Lambda(r))M_{\mu\nu}x_\nu,\end{aligned} \tag{7.4}$$

where

$$a_\Lambda(r) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{ikx}\hat{a}(|k|)\hat{f}_\Lambda(|k|), \tag{7.5}$$

$$h_\Lambda(r)x_\nu = -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{ikx}\hat{a}(|k|)\partial_\nu \hat{f}_\Lambda(|k|). \tag{7.6}$$

If A_μ supports a zero mode then $a_\Lambda(r) \underset{r\to\infty}{\sim} \nu/r^2$ since $\hat{f}_\Lambda(|k|) = 1$ for $k^2 \leq \Lambda^2$. Hence, the only result of substituting A_μ^Λ for A_μ is to replace a with $a_\Lambda + h_\Lambda$.

In Sec.V E the profile function $a(r)$ in (5.40) has a discontinuous second derivative at r=R. So $a(r)$ for $r \leq R$ would have to be smoothed to accommodate a reqularized potential. This does not in any way modify the conclusion of Sec.V E, namely that the formalism of Secs.V C and D can be implemented.

In Sec.VI B we can not choose $F_{\mu\nu}^\Lambda \in \mathrm{C}_0^\infty$ as we did for $F_{\mu\nu}$. For suppose $F_{\mu\nu}^\Lambda \in \mathrm{C}_0^\infty$. Then $\hat{F}_{\mu\nu}^\Lambda(k)$ is an entire analytic function of k_μ [40]. Therefore, we cannot set $\hat{F}_{\mu\nu}^\Lambda(k) = \hat{f}_\Lambda(|k|)\hat{F}_{\mu\nu}(k)$ since $\hat{f}_\Lambda(|k|)$ is not an entire analytic function of $|k|$. Nevertheless, $F_{\mu\nu}^\Lambda(x) = f_\Lambda$ $^*F_{\mu\nu}(x)$ is a polynomial bounded C^∞ function by Theorem IX.4 in [40]. We are now free to choose a $F_{\mu\nu} \in \mathscr{S}'$ to make $F_{\mu\nu}^\Lambda(x)$ fall off arbitrarily rapidly for $|x| > R$. So $F_{\mu\nu}^\Lambda$ can be chosen arbitrarily close to a compactly supported field. This should not change our conclusion (6.42) about the bound on I_3 for $e >> 1$.

Finally, a volume cutoff must be introduced in $\det_{\mathrm{ren}}$ -and only $\det_{\mathrm{ren}}$- in order to regularize the vacuum-vacuum amplitude Z in (2.4). As $\det_{\mathrm{ren}}$ is gauge invariant this can be done by letting $F_{\mu\nu}^\Lambda \to \mathrm{g}F_{\mu\nu}^\Lambda$, where g is a space cutoff such as $\mathrm{g} \in \mathrm{C}_0^\infty$ or $\mathrm{g} = \chi_\Gamma$, the characteristic function of a bounded region $\Gamma \subset \mathbb{R}^4$. This way of introducing g preserves the gauge invariance of $\det_{\mathrm{ren}}$.

The regularization procedure used here is a generalization of that used in the two-dimensional Yukawa model [41]. The main conclusions in this paper obtained without regulators remain valid. Thus, in (6.44)-(6.46) it is only required to replace $F_{\mu\nu}$ with $\mathrm{g}F_{\mu\nu}^\Lambda$, which is a special case of the general substitution $\det_{\mathrm{ren}}(F_{\mu\nu}) \to \det_{\mathrm{ren}}(\mathrm{g}F_{\mu\nu}^\Lambda)$. $\mathscr{F}$ is the amplitude of $F_{\mu\nu}^\Lambda$ whose scale is set by the amplitude of the underlying potential $A_\mu \in \mathscr{S}'$. It does not matter when the regulators are introduced as long as they are in place when $\det_{\mathrm{ren}}$ is inserted into (2.5) .

Interpretation of (6.44)-(6.46): Each term in representation (3.9) for $\det_{\mathrm{ren}}$ is gauge invariant and ultraviolet finite. Therefore, each term is independent of the others. It is noted in (6.44)-(6.46), with $F_{\mu\nu}$ replaced by $F_{\mu\nu}^\Lambda$ before introducing g, that $F_{\mu\nu}^\Lambda$ must be square integrable. Within the class of potentials with falloff at infinity those that support a zero mode decrease as $1/|x|$ as far as presently known. This is incompatible with $F_{\mu\nu}^\Lambda \in \mathrm{L}^2$. The terms in (6.44)-(6.46) depending on $||F^\Lambda||^2$ come from the first and third terms of (3.9). These terms were dealt with in Secs. IV and VI where it was assumed that $F_{\mu\nu}^\Lambda \in \bigcap_{r\geq 2} L^r$. Zero modes reside solely in the second term of (3.9). As shown in Sec. V it can be defined for $F_{\mu\nu}^\Lambda \in \bigcap_{r>2} L^r$. So the two terms in (6.45) and (6.46) are separately defined, each subject to its foregoing field restriction.

To regulate Z in (2.4) a volume cutoff is inserted into $\det_{\mathrm{ren}}$ as described above. When zero mode supporting potentials are introduced into $\det_{\mathrm{ren}}$ by the Maxwell measure $d\mu_0(A)$ the terms depending on $||F^\Lambda||^2$ now remain finite. Therefore, the interpretation of (6.44)-(6.46) is that they represent the asymptotic *form* of $\det_{\mathrm{ren}}$ before volume cutoffs are introduced.

For (6.44)-(6.46) to be relevant the unregularized random connections A_μ, including their assumed falloff at infinity, should have μ_0 measure one. As far as the author knows all known results for the growth at infinity of a set of random fields with measure one are for a Gaussian process whose covariance corresponds to a massive scalar field (see, for example, [52, 53]). The covariance (2.3) in a general covariant gauge does not include an infrared cutoff photon mass as none is required. To the author's knowledge, then, the behavior at infinity of a set of random Euclidean QED_4 connections with μ_0 measure one is still not settled.

VIII. CONCLUSION

Representations (2.6) and (3.9) for the Euclidean fermion determinant in QED, $\ln\det_{\mathrm{ren}}$, have been obtained that reflect its competing magnetic properties of diamagnetism and paramagnetism. This way of viewing $\ln\det_{\mathrm{ren}}$ arises since in Euclidean space $F_{\mu\nu}(x)$ may be regarded as a static, four-dimensional magnetic field. This decomposition of $\ln\det_{\mathrm{ren}}$ has the advantage of simplify-

ing its strong coupling, large field amplitude analysis for a class of random potentials/fields. The analysis is made possible by a number of theorems developed in the 1970s and 80s that are applicable to field-theoretic operators in the presence of external gauge fields.

The main results are summarized by (6.44)-(6.46) and are interpreted at the end of Sec. VII. Result (6.44) for the fast growth of $\ln \det_{\text{ren}}$ for large field variations raises doubt on whether it is integrable with any Gaussian measure whose support does not include zero mode supporting potentials. Results (6.45) and (6.46) indicate that the growth of $\ln \det_{\text{ren}}$ is slowed down or stopped by including zero mode supporting potentials in the Gaussian measure $\mathrm{d}\mu_0(A)$ introduced in Sec.II. This is *prima face* evidence that zero mode supporting potentials are necessary for the non-perturbative quantization of QED. See [54] for an earlier discussion of the non-perturbative quantization of QED.

Refer back to one of the electroweak fermion determinants such as the first one in (1.1). Suppose after being properly defined its large amplitude Maxwell field variation coincides with that of $\ln \det_{\text{ren}}$. Then (6.45) and (6.46) provide *prima face* evidence that the non-perturbative quantization of the electroweak model also requires its Maxwell Gaussian measure to have support from zero mode supporting potentials. This assumes that the Maxwell field integration follows next after integrating out the fermion degrees of freedom.

Given such Gaussian measures are they such that no measurable subset of potentials results in the fast growing charge renormalization term in (6.45) and (6.46) becoming dominant? This is entering unknown territory that needs to be explored.

If the QED determinant grows faster than a quadratic in the Maxwell field for a measurable set of fields then there may be a connection between this and the photon propagator's Landau pole [5, 56]. The precise connection, if any, remains to be worked out.

It might be objected that the non-perturbative quantization of the electroweak model is irrelevant since perturbative expansions appear to be adequate at presently available energies. This opinion neglects the fact that the electroweak model is not asymptotically free. At some point the model's non-perturbative content will be required.

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Appendix A: $\Delta_A^{1/2}\,\sigma F\,\Delta_A^{1/2}$

It is claimed that $\Delta_A^{1/2}\,\sigma F\,\Delta_A^{1/2}$ belongs to the trace ideal $\mathscr{I}_3$ for $F_{\mu\nu} \in \cap_{q>2} L^q(\mathbb{R}^4)$. The trace ideal $\mathscr{I}_p(1 \leq p < \infty)$ is defined as those compact operators A with $\|A\|_p^p = \mathrm{Tr}\,((A^\dagger A)^{p/2}) < \infty$. General properties of $\mathscr{I}_p$ spaces used here may be found in [8–10]. To simplify notation we set $e = 1$ in this appendix.

To decide whether $\Delta_A^{1/2}\,\sigma F\,\Delta_A^{1/2} \in \mathscr{I}_3$ it suffices to deal with $\Delta_A^{1/2}\,|F|\,\Delta_A^{1/2}$ ($F_{\mu\nu}^2 = |F|^2$) since $\sigma F/|F|$ is unitary. Then $\Delta_A^{1/2}\,|F|\,\Delta_A^{1/2} \in \mathscr{I}_3$ if $|F|^{1/2}\,\Delta_A^{1/2} \in \mathscr{I}_6$ since by Hölder's inequality for $\mathscr{I}_p$ spaces

$$\left\|\Delta_A^{1/2}\,|F|\,\Delta_A^{1/2}\right\|_3 \leq \left\|\Delta_A^{1/2}\,|F|^{1/2}\right\|_6 \left\||F|^{1/2}\,\Delta_A^{1/2}\right\|_6. \tag{A1}$$

If $|F|^{1/2}\,\Delta_A^{1/2} \in \mathscr{I}_6$ then so does its adjoint $\Delta_A^{1/2}\,|F|^{1/2}$ by the general properties of $\mathscr{I}_p$ spaces. Then

$$\begin{aligned}\left\||F|^{1/2}\,\Delta_A^{1/2}\right\|_6^6 &= \mathrm{Tr}\left(\Delta_A^{1/2}\,|F|\Delta_A|F|\Delta_A|F|\,\Delta_A^{1/2}\right)\\ &\leq \mathrm{Tr}\left(\Delta^{1/2}|F|\Delta|F|\Delta|F|\Delta^{1/2}\right)\\ &= \left\||F|^{1/2}\Delta^{1/2}\right\|_6^6.\end{aligned} \tag{A2}$$

The first line of (A2) may be written in coordinate space. Then the second line follows from Kato's inequality in the form [12–14, 33, 41–46]

$$|\Delta_A(x,y)| \leq \Delta(x-y), \tag{A3}$$

where $\Delta(x) = mK_1(mx)/(4\pi^2 x)$, and K_1 is a modified Bessel function. We also made use of the identity

$$\Delta_A^{1/2}\,(x,y) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}a}{\sqrt{a}}\,\langle x|\,\frac{1}{(P-A)^2+m^2+a}\,|y\rangle\,, \tag{A4}$$

to obtain $\left|\Delta_A^{1/2}\,(x,y)\right| < \Delta^{1/2}(x-y)$ from (A3) with $\Delta^{1/2}(x) = (m/(2\pi^{5/3}x))^{3/2}K_{3/2}(mx)$. This result for $\Delta^{1/2}$ is obtained from representation (A4) with $A_\mu = 0$ using integral 2.16.3.8 of [35].

To prove that $|F|^{1/2}\Delta^{1/2} \in \mathscr{I}_6$ it has to be shown that this operator maps $L^2(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$ for $F_{\mu\nu} \in \cap_{q>2} L^2(\mathbb{R}^4)$. Let $\varphi = |F|^{1/2}\,\Delta_A^{1/2}\,\psi$, $\psi \in L^2$. Then by Kato's inequality

$$\begin{aligned}\|\varphi\|_2^2 &= \int \psi^*\,\Delta_A^{1/2}\,|F|\,\Delta_A^{1/2}\,\psi\\ &\leq \int |\psi|\Delta^{1/2}|F|\Delta^{1/2}|\psi|.\end{aligned} \tag{A5}$$

Let $\rho(x) = \int \mathrm{d}^4y\,\Delta^{1/2}(x-y)|\psi(y)| = \Delta^{1/2} \star |\psi|(x)$. By Hölder's inequality

$$\|\varphi\|_2 \leq \||F|^{1/2}\rho\|_2 \leq \|\rho\|_p \||F|^{1/2}\|_q, \tag{A6}$$

where $1/p + 1/q = 1/2$, p, $q \geq 1$. Since we assume $q > 4$ in (A6) then $1 \leq p < 4$. Use Young's inequality in the form given in Table IX.1 of [40],

$$\|f \star g\|_r \leq \|f\|_s \|g\|_t, \tag{A7}$$

with $1/s + 1/t = 1 + 1/r$, $1 \le r, s, t \le \infty$. Then $\|\rho\|_p = \|\Delta^{1/2} \star |\psi|\|_p \le \|\Delta^{1/2}\|_r \|\psi\|_2$, $r < 4/3$. As $\Delta^{1/2}(x)$ behaves as $1/x^3$ for $x \to 0$ and exponentially decreases for $x \to \infty$ then $\|\Delta^{1/2}\|_r < \infty$, proving that $\varphi \in L^2$.

To complete the proof that $|F|^{1/2}\Delta^{1/2} \in \mathscr{I}_6$ we rely on the following theorem specialized to four dimensions [8, 47].

Theorem A: Let $f(x)g(-i\nabla)$ map $L^2(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.

If $f \in L^r(\mathrm{d}^4x)$ and $g \in L^r(\mathrm{d}^4p)$ with $2 \le r < \infty$, then $f(x)g(-i\nabla)$ is in $\mathscr{I}_r$ and

$$\|f(x)g(-i\nabla)\|_{\mathscr{I}_r} \le (2\pi)^{-4/r}\|f\|_{L^r}\|g\|_{L^r}. \tag{A8}$$

We have just shown that $|F|^{1/2}\Delta^{1/2}$ is a bounded operator on $L^2(\mathbb{R}^4)$, for $F_{\mu\nu} \in \cap_{q>2} L^q(\mathbb{R}^4)$. By inspection $|F|^{1/2} \in L^6(\mathrm{d}^4x)$ and $(p^2+m^2)^{-1/2} \in L^6(\mathrm{d}^4p)$ and hence $|F|^{1/2}\Delta^{1/2} \in \mathscr{I}_6$. This establishes that $\Delta_A^{1/2}\,|F|\,\Delta_A^{1/2} \in \mathscr{I}_3$ on referring to (A1) and (A2), and hence so does $\Delta_A^{1/2}\,\sigma F\,\Delta_A^{1/2}$.

Finally, in both Sec.VB and Appendix D it is claimed that if $\varphi \in L^2$ then so does $\psi = \Delta_A^{1/2}\,\varphi$. We have

$$|\psi(x)| \le \int \mathrm{d}^4 \left|\Delta_A^{1/2}(x,y)\right| |\varphi(y)| \tag{A9}$$

$$\le \int \mathrm{d}^4y\, \Delta^{1/2}(x-y)|\varphi(y)| = \Delta^{1/2} \star |\varphi|(x). \tag{A10}$$

Then by Young's inequality (A7), $\|\psi\|_2 \le \|\Delta^{1/2} \star |\varphi|\|_2 \le \|\Delta^{1/2}\|_1 \|\varphi\|_2 < \infty$ since $\|\Delta^{1/2}\|_1 < \infty$.

Appendix B: Equivalence of the two sides of Eq. (3.6)

Reduce notation by setting $B = \frac{1}{2}\sigma\mathrm{F}$ and $e = 1$. Begin with the right-hand side of (3.6) by substituting (3.5) and obtain

$$\begin{aligned}\mathrm{RHS} = \int_0^\infty \frac{\mathrm{d}t}{t} e^{-tm^2}\mathrm{Tr}\Big(&e^{-tD^2} - e^{-t(D^2+B)} - \int_0^t \mathrm{d}t e^{-(t-s)D^2} B e^{-sD^2} \\ &+ \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 e^{-(t-s_1-s_2)D^2} B e^{-s_2D^2} B e^{-s_1D^2}\Big).\end{aligned} \tag{B1}$$

Eliminate the O(B) term by taking the spin trace of this term. Then

$$\begin{aligned}\frac{\mathrm{d(RHS)}}{\mathrm{d}m^2} =&\mathrm{Tr}\left[\left(D^2+B+m^2\right)^{-1} - \left(D^2+m^2\right)^{-1}\right] \\ &- \int_0^\infty \mathrm{d}t e^{-tm^2}\mathrm{Tr}\left[\int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 e^{-(t-s_1-s_2)D^2} B e^{-s_2D^2} B e^{-s_1D^2}\right].\end{aligned} \tag{B2}$$

Note that

$$\begin{aligned}\left(D^2+B+m^2\right)^{-1} - \left(D^2+m^2\right)^{-1} =& -\frac{1}{D^2+m^2}B\frac{1}{D^2+m^2} + \frac{1}{D^2+m^2}B\frac{1}{D^2+m^2}B\frac{1}{D^2+m^2} \\ &- \frac{1}{D^2+B+m^2}B\frac{1}{D^2+m^2}B\frac{1}{D^2+m^2}B\frac{1}{D^2+m^2}.\end{aligned} \tag{B3}$$

Substitute (B3) in (B2) and eliminate the O(B) term by tracing over its spin to get

$$\begin{aligned}\frac{\mathrm{d(RHS)}}{\mathrm{d}m^2} =&\mathrm{Tr}(R) + \mathrm{Tr}\left(\Delta_A B \Delta_A B \Delta_A\right) \\ &- \int_0^\infty \mathrm{d}t e^{-tm^2}\mathrm{Tr}\left[\int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 e^{-(t-s_1-s_2)D^2} B e^{-s_2D^2} B e^{-s_1D^2}\right],\end{aligned} \tag{B4}$$

where

$$R = -\frac{1}{D^2+B+m^2} B\Delta_A B\Delta_A B\Delta_A. \tag{B5}$$

The trace of R is obviously finite. The second trace in (B4) is cancelled by the last integral. To see this use the cyclic property of the trace in the last integral and integrate the s_1-integral by parts to obtain

$$\begin{aligned}\frac{\mathrm{d(RHS)}}{\mathrm{d}m^2} =&\mathrm{Tr}(R) + \mathrm{Tr}\left(\Delta_A B \Delta_A B \Delta_A\right) \\ &- \int_0^\infty \mathrm{d}t\mathrm{Tr}\left(e^{-(D^2+m^2)t}\int_0^t \mathrm{d}s s e^{sD^2} B e^{-sD^2} B\right).\end{aligned} \tag{B6}$$

The trace manipulations here and below are allowed due to the presence of the exponentiated (bounded) operators. Now integrate the t-integral by parts twice, firstly to get rid of the s-integration, and secondly to eliminate the factor t to obtain

$$\begin{aligned}\frac{\mathrm{d(RHS)}}{\mathrm{d}m^2} =&\mathrm{Tr}(R)\\ =&-\frac{1}{8}\mathrm{Tr}\left(\frac{1}{D^2+\frac{1}{2}\sigma F+m^2}\sigma F\Delta_A\sigma F\Delta_A\sigma F\Delta_A\right).\end{aligned} \tag{B7}$$

Now relate the left-hand side of (3.6) to the result (B7). We know that $T\equiv\Delta_A^{1/2}\frac{1}{2}\sigma F\Delta_A^{1/2}\in\mathscr{I}_3$. Then [11]

$$R_3(T)\equiv(1+T)e^{-T+T^2/2}-1\in\mathscr{I}_1, \tag{B8}$$

so that the the relation $\mathrm{lndet}(1+R_3)=\mathrm{Trln}(1+R_3)$ is valid. From the definition (3.7) this gives

$$\mathrm{lndet}_3(1+T)=\mathrm{Tr}\left[\ln(1+T)-T+\frac{1}{2}T^2\right]. \tag{B9}$$

Noting that

$$\frac{\mathrm{d}T}{\mathrm{d}m^2}=-\frac{1}{2}\Delta_A T-\frac{1}{2}T\Delta_A, \tag{B10}$$

differentiation of (B9) with respect to m^2 gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m^2}\mathrm{lndet}_3\left(1+\Delta_A^{1/2}\frac{1}{2}\sigma F\Delta_A^{1/2}\right)&=-\mathrm{Tr}\left(\Delta_A\frac{1}{1+T}T^3\right)\\ =&-\frac{1}{8}\mathrm{Tr}\left(\frac{1}{D^2+\frac{1}{2}\sigma F+m^2}\sigma F\frac{1}{D^2+m^2}\sigma F\frac{1}{D^2+m^2}\sigma F\frac{1}{D^2+m^2}\right)\\ =&\frac{\mathrm{d(RHS)}}{\mathrm{d}m^2}.\end{aligned} \tag{B11}$$

Since both sides of (3.6) vanish for $m=\infty$ then the two sides are equivalent on integrating (B11).

Appendix C: Simplification of Eq. (3.8)

Refer to the last term in (3.8) and take the spin trace. Denoting this term by Π it is

$$\Pi=e^2\int_0^\infty\frac{\mathrm{d}t}{t}\,e^{-tm^2}\left[\frac{\|F\|^2}{32\pi^2}-\mathrm{Tr}\int_0^t\mathrm{d}s_1\int_0^{t-s_1}\mathrm{d}s_2\,e^{-(t-s_1-s_2)D^2}F_{\mu\nu}e^{-s_2D^2}F_{\mu\nu}e^{-s_1D^2}\right]. \tag{C1}$$

To $\mathrm{O}(e^2)$ (C1) gives

$$\Pi=\frac{e^2}{32\pi^2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left|\hat{F}_{\mu\nu}(k)\right|^2\ln\left(\frac{k^2z(1-z)+m^2}{m^2}\right)+\mathrm{O}(e^4), \tag{C2}$$

verifying that Π is finite and that $\Pi(m=\infty)=0$, as inspection of (C1) indicates.

To simplify (C1) integrate the s_1-integral by parts, use the cyclic property of the trace, and let $s_1=s$ to get

$$\Pi=e^2\int_0^\infty\frac{\mathrm{d}t}{t}\,e^{-tm^2}\left[\frac{\|F\|^2}{32\pi^2}-\mathrm{Tr}\int_0^t\mathrm{d}s\,se^{-(t-s)D^2}F_{\mu\nu}e^{-sD^2}F_{\mu\nu}\right]. \tag{C3}$$

It is safe to differentiate Π with respect to m^2 as this makes (C3) even more ultraviolet convergent. Doing this and integrating the t-integral by parts gives

$$\begin{aligned}-\frac{\mathrm{d}\Pi}{\mathrm{d}m^2} &= e^2 \int_0^\infty \mathrm{d}t\, e^{-tm^2}\left[\frac{\|F\|^2}{32\pi^2} - t\,\mathrm{Tr}\left(\frac{1}{D^2+m^2}F_{\mu\nu}e^{-tD^2}F_{\mu\nu}\right)\right] \\ &= e^2 \int_0^\infty \mathrm{d}t\,\left[e^{-tm^2}\frac{\|F\|^2}{32\pi^2} + \int_0^\infty \mathrm{d}s\ \mathrm{Tr}\left(e^{-s(D^2+m^2)}F_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}m^2}e^{-t(D^2+m^2)}F_{\mu\nu}\right)\right] \\ &= e^2\frac{\mathrm{d}}{\mathrm{d}m^2}\int_0^\infty \mathrm{d}t\,\left[-e^{-tm^2}\frac{\|F\|^2}{32\pi^2 t} + \frac{1}{2}\int_0^\infty \mathrm{d}s\ \mathrm{Tr}\left(e^{-s(D^2+m^2)}F_{\mu\nu}e^{-t(D^2+m^2)}F_{\mu\nu}\right)\right].\end{aligned} \tag{C4}$$

Hence,

$$\Pi = e^2\int_0^\infty \mathrm{d}t\, e^{-tm^2}\left[\frac{\|F\|^2}{32\pi^2 t} - \frac{1}{2}\,\mathrm{Tr}\left(e^{-tD^2}F_{\mu\nu}\Delta_A F_{\mu\nu}\right)\right], \tag{C5}$$

since $\Pi(m=\infty)=0$. This is the result in (3.9).

As a check on (C5), its $\mathrm{O}(e^2)$ expansion reproduces the result (C2). In (3.9) $\det_3$ has no $\mathrm{O}(e^2)$ term by its definition, and $\ln\det_{\mathrm{SQED}}$ in (3.3) to $\mathrm{O}(e^2)$ is

$$\ln\det{}_{\mathrm{SQED}} = -\frac{e^2}{64\pi^2}\int_0^1 \mathrm{d}z\,(1-2z)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left|\hat{F}_{\mu\nu}(k)\right|^2\ln\left(\frac{k^2z(1-z)+m^2}{m^2}\right) + \mathrm{O}(e^4). \tag{C6}$$

Combining (C2) with (C6) following (3.9) gives the textbook result for the lowest-order vacuum polarization graph with on-shell renormalization:

$$\ln\det{}_{\mathrm{ren}} = \frac{e^2}{8\pi^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left|\hat{F}_{\mu\nu}(k)\right|^2\int_0^1\mathrm{d}z\,z(1-z)\ln\left(\frac{k^2z(1-z)+m^2}{m^2}\right) + \mathrm{O}(e^4). \tag{C7}$$

Appendix D: Eigenvalue pairs of $\Delta_A^{1/2}\sigma F\Delta_A^{1/2}$

From the equation for the scalar field propagator in the external potential A_μ,

$$\left[\left(\frac{1}{i}\partial_\mu - eA_\mu\right)^2 + m^2\right]\Delta_A(x,y) = \delta(x-y), \tag{D1}$$

obtain by inspection

$$\Delta_{A+\partial\lambda}(x,y) = e^{ie(\lambda(x)-\lambda(y))}\Delta_A(x,y). \tag{D2}$$

Referring to the representation (A4) of $\Delta_A^{1/2}$ conclude that it transforms under $A\to A+\partial\lambda$ in the same way as Δ_A. Therefore, it is evident that $\det_3(1+\Delta_A^{1/2}\frac{e}{2}\sigma F\Delta_A^{1/2})$ is gauge invariant.

Noting (D2), define the gauge invariant propagator

$$\tilde{\Delta}_A(x,y) = e^{-ie\int_y^x \mathrm{d}\xi^\mu A_\mu(\xi)}\Delta_A(x,y). \tag{D3}$$

In what follows it is not necessary to specify the line integral's path. Taking the complex conjugate of (D1) deduce that $\Delta_A^* = \Delta_{-A}$ and hence from (D3) that

$$\tilde{\Delta}_A^*(x,y) = \tilde{\Delta}_{-A}(x,y). \tag{D4}$$

Refer to (5.7) and consider an eigenstate φ of $\frac{e}{2}\Delta_A^{1/2}\sigma F\Delta_A^{1/2}$ with eigenvalue $-\lambda$. Let $\psi = \Delta_A^{1/2}\varphi$. Then

$$\frac{e}{2}\Delta_A\sigma F\psi = -\lambda\psi. \tag{D5}$$

Since $\varphi\in\mathrm{L}^2$ so does ψ as shown at the end of Appendix A. We will now show that there is an eigenstate ψ_C with eigenvalue λ.

Substitute (D3) in (D5):

$$\begin{aligned}&\frac{e}{2}\int \mathrm{d}^4y\,\tilde{\Delta}_A(x,y)\sigma F(y)e^{-ie\int_z^y \mathrm{d}\xi^\mu A_\mu(\xi)}\psi(y) \\ &\qquad = -\lambda e^{-ie\int_z^x \mathrm{d}\xi^\mu A_\mu(\xi)}\psi(x),\end{aligned} \tag{D6}$$

where z is an arbitrary point in $\mathbb{R}^4$. On taking the complex conjugate of (D6) we seek a matrix C such that

$C\gamma_\mu^* C^{-1} = -\gamma_\mu$. In the representation

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma_0 = -i\begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2, \end{pmatrix}, \tag{D7}$$

one may choose $C = \gamma_3\gamma_1$. Since $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/(2i)$, $C\sigma^* C^{-1} = -\sigma$. Substitution of this result into the complex conjugate of (D6) gives together with (D4),

$$\frac{e}{2}\int \mathrm{d}^4y \tilde{\Delta}_{-A}(x,y)\sigma F(y) e^{ie\int_z^y \mathrm{d}\xi^\mu A_\mu} C\psi^*(y) = +\lambda e^{ie\int_z^x \mathrm{d}\xi^\mu A_\mu} C\psi^*(x). \tag{D8}$$

$\tilde{\Delta}_A$ is gauge invariant and depends only on $F_{\mu\nu}$ and invariants derived from it. The second line in (5.2) when expanded in powers of T consists of loops with $\tilde{\Delta}_A$ between insertions of σF as the phase factors from Δ_A cancel in the trace. Since $\ln\det_3$ is real, $\tilde{\Delta}_A$ is real and hence by (D4) $\tilde{\Delta}_{-A} = \tilde{\Delta}_A$, expressing C-invariance. Inserting this result in (D8) we conclude that for each eigenstate ψ of $\frac{e}{2}\Delta_A \sigma F$ with eigenvalue $-\lambda$ there is a paired eigenstate

$$\psi_C(x) = e^{2ie\int_z^x \mathrm{d}\xi^\mu A_\mu(\xi)} C\psi^*(x), \tag{D9}$$

with eigenvalue $+\lambda$.

Appendix E: Calculation of λ

Substitute either of the expansions (5.46), (5.47) or (5.48), (5.49) to O(m^2) in (5.42) and obtain using (5.50)

$$f_2'' + \left(\frac{4j+3}{r} - 2era\right) f_2' = 1 - 4ea\delta_2 - er\frac{\mathrm{d}a}{\mathrm{d}r}\delta_2. \tag{E1}$$

The solution of (E1) at $r = R$ that is finite at $r = 0$ is

$$f_2'(R) = \int_0^R \mathrm{d}r \left(\frac{r}{R}\right)^{4j+3} (1 - 4e\delta_2 a(r) - e\delta_2 r a'(r))\, e^{2e\int_r^R \mathrm{d}s s a(s)}. \tag{E2}$$

To $O(m^2)$ the boundary condition (5.45) requires

$$Rf_2'(R) = \frac{R^2/2}{2j+2-e\nu} + \frac{e\nu\delta_2}{e\nu - 2j - 1}. \tag{E3}$$

Note that $a(r)$, regardless of the sign of C in (5.40), approaches ν/r^2 as $r \nearrow R$. Therefore, $f_2'(R)$ in (E2) is exponentially increasing with e while the right-hand side of (E3) has no such exponential growth. Accordingly, the boundary condition (E3) requires δ_2 to satisfy

$$\delta_2 = \frac{\int_0^R \mathrm{d}r \left(\frac{r}{R}\right)^{4j+3} e^{2e\int_r^R \mathrm{d}s s a}}{e\int_0^R \mathrm{d}r \left(\frac{r}{R}\right)^{4j+3} (4a + ra')\, e^{2e\int_r^R \mathrm{d}s s a}} + c, \tag{E4}$$

where c is an exponentially decaying function of e. Insert (E2) in (E3) and then refer to (E4) to obtain an equation for c:

$$ceR\int_0^R \mathrm{d}r \left(\frac{r}{R}\right)^{4j+3} (4a + ra')\, e^{2e\int_r^R \mathrm{d}s s a} = \frac{R^2/2}{e\nu - 2j - 2} + \frac{e\nu\delta_2}{2j+1-e\nu}. \tag{E5}$$

As δ_2 is determined by (E4) up to an exponentially decaying term, (E5) is sufficient to determine c.

It remains to estimate δ_2 in (E4) with $e\nu > 2j+2$ and $e \gg 1$. The structure of the first term in (E4) suggests

Laplace's method [16] as the most direct way of proceeding. Consider the numerator of (E4):

$$I=\frac{1}{R}\int_0^R \mathrm{d}r\left(\frac{r}{R}\right)^{4j+3} e^{2e\int_r^R \mathrm{d}s s a}. \tag{E6}$$

Let $r=xR$, $s=tR$ and set

$$g(x)=(4j+3)\ln(x)+2eR^2\int_x^1 \mathrm{d}t t a. \tag{E7}$$

Let $g'(x_0)=0$. Since $e\nu>2j+2$, $g'(1)<0$ and $g'(x)\to\infty$ for $x\searrow 0$ then $g''(x_0)<0$. Hence, $0<x_0<1$. For any sign of C in (5.40) and $\epsilon\geq 2$ a sketch of $(4j+3)/x$ and $2eR^2xa$ versus x indicates that $4a(x_0)+x_0a'(x_0)>0$. These strong statements can be made due to the simplicity of a in (5.40). Therefore, for $e\gg 1$

$$I=e^{g(x_0)}\sqrt{\frac{2\pi}{|g''(x_0)|}}\left(1+O(g^{i\nu}(x_0)/e^2)\right). \tag{E8}$$

Since $a(r)$ is a smooth function for $0<r<R$, $g^{i\nu}(x_0)$ is finite and $O(e)$ or less. Repeating this procedure for the denominator of (E4) gives for $e\nu>2j+2$, $e\gg 1$

$$\delta_2=\frac{1/e}{4a(r_0)+r_0a'(r_0)}\left(1+O(1/e)\right)>0, \tag{E9}$$

where $r_0=Rx_0$ is the unique root in the interval $0<r<R$ of

$$4j+3-2er^2a(r)=0. \tag{E10}$$

Refer to (5.14) and define the spin trace norm of an operator A by $\|A\|_1=\mathrm{Tr}\left(A^\dagger A\right)^{1/2}$ so that $\frac{1}{2}\|(\sigma F)^+\|_1=|4a+ra'|$, where $(\sigma F)^+$ is defined by (5.14). Then (E9) becomes

$$\delta_2=\frac{2}{e\|(\sigma F(r_0))^+\|_1}\left(1+O(1/e)\right). \tag{E11}$$

Here $F_{\mu\nu}(r_0)$ is a smoothly varying function on $0<r_0<R$ and hence slowly varying for $j=0,1/2,.,j_{max}$ and $e\nu>2j+2$, $e\gg 1$.

Repeated application of Laplace's method gives the following additional results for $e\gg 1$. For $j=0,1/2,.,j_{max}-1/2$, $e\nu>2j+2$ with $e\nu=N+\Delta$, $0<\Delta<1$, $N=2,3,.,j_{max}=(N-2)/2$, δ_4 in (5.47) is

$$\delta_4=-\delta_2^2+O\left(\frac{R^4}{e^4}\right). \tag{E12}$$

For $j=j_{max}$, $\delta_{2\alpha_0}$ in (5.49) exponentially decreases with e and the $O(m^4)$ term is the same as that in (5.47) with δ_4 given by (E12). For $e\nu=3,4...$ and $j=j_{max}=(N-3)/2$, (5.47) holds with δ_2, δ_4 given by (E11) and (E12).

Appendix F: Zero mass limit of $\det_{\mathrm{ren}}$

The renormalized determinant in (2.1) may be equivalently expressed as [7, 31, 48]

$$\det{}_{ren}\left(1-eS\not{A}\right)=\exp\left(\Pi_2+\Pi_3+\Pi_4\right)\det{}_5\left(1-eS\not{A}\right), \tag{F1}$$

where

$$\ln\det{}_5\left(1-eS\not{A}\right)=\mathrm{Tr}\left[\ln\left(1-eS\not{A}\right)+\sum_{n=1}^{4}\left(eS\not{A}\right)^n/n\right]. \tag{F2}$$

As evident from (F1), $\det_5$ is the remainder of $\det(1-eS\not{A})$ after the $O(e^2,e^3,e^4)$ graphs Π_2,Π_3 and Π_4 have been factored out. To maintain equality with (2.1) they are defined by the power series expansion of its right-hand side to $O(e^4)$. This definition gives the on-shell subtracted vacuum polarization graph Π_2 in (C7); $\Pi_3=0$, and the gauge invariant photon-photon scattering graph Π_4. A Hilbert space can be found on which $S\not{A}$ is a compact operator belonging to $\mathscr{I}_r$, $r>4$ provided $A_\mu\in\bigcap_{r\geq 4+\epsilon}L^r$ [7, 31, 48]. The trace ideal $\mathscr{I}_r$ is discussed in Sec.III and Appendix A. Then $S\not{A}\in\mathscr{I}_5$ since $\mathscr{I}_{4+\epsilon}\subset\mathscr{I}_5$, and hence $\det_5$ is an entire function of e of order 4 [14]. It has no zeros for real e, and since $\det_{\mathrm{ren}}(e=0)=1$, $\det_{\mathrm{ren}}>0$ for all real e. It will now be shown that the $m=0$ limit of $\det_{\mathrm{ren}}$ is finite when Π_2 is subtracted off-shell, provided $A_\mu\in\bigcap_{r\geq 4-\epsilon}L^r(\mathbb{R}^4)$, $\epsilon>0$. This excludes zero-mode supporting potentials that fall off as $1/x$ and which induce divergent mass singularities in $\ln\det_{\mathrm{ren}}$ [24, 49, 50]. Our analysis of the $m=0$ limit of $\det_{\mathrm{ren}}$ is a generalization of that in [31] for massless QED_2.

Instead of dealing with the operator $S\not{A}$ at $m=0$ we make a similarity transformation that leaves $\det_5$ invariant. Setting $m=0$ let

$$S\not{A}\to\frac{\not{p}}{|p|}\frac{1}{|p|^{1/2}}|A|^{1/2}\frac{\not{A}}{|A|}|A|^{1/2}\frac{1}{|p|^{1/2}}, \tag{F3}$$

where $|A|=(A_\mu^2)^{1/2}$. Because $\not{p}/|p|$ and $\not{A}/|A|$ are unitary it suffices to consider the operator $K=|p|^{-1/2}|A||p|^{-1/2}$. We claim that $K\in\mathscr{I}_r$, $r>4$ provided $A_\mu\in\bigcap_{q\geq 4-\epsilon}L^q(\mathbb{R}^4),\epsilon>0$. If $K\in\mathscr{I}_r$ then by Hölder's inequality for $\mathscr{I}_r$ spaces

$$\|K\|_r\leq\left\|\frac{1}{|p|^{1/2}}|A|^{1/2}\right\|_s\left\||A|^{1/2}\frac{1}{|p|^{1/2}}\right\|_s, \tag{F4}$$

with $s=2r>8$. If $|A|^{1/2}|p|^{-1/2}\in\mathscr{I}_s$ then so does its adjoint $|p|^{-1/2}|A|^{1/2}$ by the general properties of $\mathscr{I}_p$ spaces. Let

$$B=|A|^{1/2}\frac{1}{|p|^{1/2}}=B_1+B_2, \tag{F5}$$

where

$$B_1 = |A|^{1/2}\left(\frac{1}{|p|^{1/2}} - \frac{1}{(p^2+\mu^2)^{1/4}}\right), \qquad \text{(F6)}$$

$$B_2 = |A|^{1/2}\,\frac{1}{(p^2+\mu^2)^{1/4}}, \qquad \text{(F7)}$$

and where μ^2 is an arbitrary mass parameter. To prove that B_1, $B_2 \in \mathscr{I}_s$, $s > 8$, it has to be first shown that these operators map $L^2(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.

We begin with B_1. Let $g_1 = \Delta_1 * f$, $f \in L^2$, where

$$\Delta_1(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ipx}\left(\frac{1}{|p|^{1/2}} - \frac{1}{(p^2+\mu^2)^{1/4}}\right). \qquad \text{(F8)}$$

Then $\Delta_1(x)$ behaves as $\mu^2/x^{3/2}$ for $x \to 0$ and $1/x^{7/2}$ for $x \to \infty$. Let $h_1 = |A|^{1/2}\, g_1$. By Hölder's inequality

$$\|h_1\|_2 = \left\||A|^{1/2}\, g_1\right\|_2 \le \left\||A|^{1/2}\right\|_p \|g_1\|_q, \qquad \text{(F9)}$$

with $1/p + 1/q = 1/2$, $p, q \ge 1$. By Young's inequality (A7), $\|g_1\|_q = \|\Delta_1 * f\|_q \le \|\Delta_1\|_r \|f\|_2$ with $1/q + 1/2 = 1/r$, $q, r \ge 1$. Referring to the properties of Δ_1 it is evident that $\|\Delta_1\|_r < \infty$ provided $8/7 < r < 8/3$. Choose $q > 8/3$. From $1/p + 1/q = 1/2$ obtain $p < 8$. Then (F9) allows $A_\mu \in \bigcap_{p \ge 4-\epsilon} L^p$, $\epsilon > 0$. Under this condition $\|h_1\|_2 < \infty$ and hence B_1 is an operator on L^2.

Next consider B_2. The Fourier transform of $(p^2 + \mu^2)^{-1/4}$ in four dimensions is undefined. So consider

$$g_2(x) = \Delta_2 * f(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ipx} \frac{\hat{f}(p)}{(p^2+\mu^2)^{1/4}}. \qquad \text{(F10)}$$

Since $\|f\|_2 = \|\hat{f}\|_2/(2\pi)^2$ then $\hat{f}(p)$ behaves as $1/p^{2+\epsilon}$ for $p \to \infty$ and $1/p^{2-\epsilon}$ for $p \to 0$. Therefore, $g_2(x)$ behaves as $1/x^{2+\epsilon}$ for $x \to \infty$ and $1/x^{3/2-\epsilon}$ for $x \to 0$. Then B_2 maps L^2 into L^2 since $h_2 = |A|^{1/2}\, g_2$ satisfies $\|h_2\|_2 = \left\||A|^{1/2}\, g_2\right\|_2 \le \left\||A|^{1/2}\right\|_p \|g_2\|_q$ with $1/p + 1/q = 1/2$, $p, q \ge 1$. Thus $\|g_2\|_q < \infty$ for $2 < q < 8/3$ which requires $p > 8$ or $A_\mu \in \bigcap_{p>4} L^p$.

To complete that proof that B_1 , $B_2 \in \mathscr{I}_s$, $s > 8$ we rely on Theorem A in Appendix A. For B_1, since

$$\begin{aligned}\frac{1}{|p|^{1/2}} - \frac{1}{(p^2+\mu^2)^{1/4}} &= O\left(|p|^{-3/2}\right), |p| \to \infty \\ &= O\left(|p|^{-1/2}\right), |p| \to 0, \qquad \text{(F11)}\end{aligned}$$

the left-hand side belongs to $L^s(\mathrm{d}^4 p)$ for $8/3 < s < 8$. It was just shown that B_1 is a bounded operator on L^2 if $|A|^{1/2} \in L^s(\mathrm{d}^4 x)$, $s < 8$. By Theorem A $B_1 \in \mathscr{I}_s$, $s < 8$, and therefore by the general properties of $\mathscr{I}_p$ spaces, $B_1 \in \mathscr{I}_s$, $8 \le s \le \infty$.

For B_2 evidently $(p^2 + \mu^2)^{-1/4} \in L^s(\mathrm{d}^4 p)$ for $s > 8$. For B_2 to be a bounded operator on L^2 it was found that $|A|^{1/2} \in L^s(\mathrm{d}^4 x), s > 8$. Hence $B_2 \in \mathscr{I}_s$, $8 < s \le \infty$ by Theorem A.

It has now been established that $B_1 + B_2 = |A|^{1/2}\, |p|^{-1/2} \in \mathscr{I}_s$, $8 < s \le \infty$ provided $A_\mu \in \bigcap_{r \ge 4-\epsilon} L^r$, $\epsilon > 0$. Referring to (F4), $K = |p|^{-1/2}\, |A|\, |p|^{-1/2} \in \mathscr{I}_r, 4 < r \le \infty$, and hence $\det_5$ is well-defined at $m = 0$ since $K \in \mathscr{I}_5$. The loop expansion of $\det_5$ makes sense, and so the similarity transformation defined in (F3) is valid, allowing us to conclude that $S\not{A}|_{m=0} \in \mathscr{I}_5$ for the restricted class of A_μ potentials considered here.

It remains to demonstrate the continuity of the $m = 0$ limit of $\det_5(1 - eS\not{A}) = \det_5(1 - e\not{A}S)$ for $m > 0$. We will deal with the operator $\not{A}S$. The continuity of the $m = 0$ limit of $\det_5$ will follow from a theorem Gohberg and Kreĭn, Ch. 4, Th. 2.1 [10]: Let $A \in \mathscr{I}_p$, where p is a positive integer, and let F be an arbitrary closed bounded set. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any operator $B \in \mathscr{I}_p$,

$$\max_{\mu \in F} |\det{}_p\,(1 - \mu A) - \det{}_p\,(1 - \mu B)| < \epsilon$$

whenever $\|A - B\|_p < \delta$.
Consider

$$\not{A}S - \not{A}S_{m=0} = \not{A}\frac{m^2 \not{p}}{p^2(p^2+m^2)} + \not{A}\frac{m}{p^2+m^2}. \qquad \text{(F12)}$$

It is now known that $\not{A}S$, $\not{A}S_{m=0} \in \mathscr{I}_5$ for $A_\mu \in \bigcap_{r \ge 4-\epsilon} L^r$, $\epsilon > 0$. Then

$$\left\|\not{A}S - \not{A}S_{m=0}\right\|_5 \le \left\|\not{A}\frac{m^2 \not{p}}{p^2(p^2+m^2)}\right\|_5 + \left\|\not{A}\frac{m}{p^2+m^2}\right\|_5. \qquad \text{(F13)}$$

Let

$$B_3 = \not{A}\frac{\not{p}}{p^2(p^2+m^2)}, \qquad \text{(F14)}$$

where B_3 is an operator on L^2 for A_μ restricted as above. The proof of this proceeds in exactly the same way as in the case of B_1 above. The form of B_3 allows immediate application of Theorem A, Appendix A. By inspection $\not{p}/[p^2(p^2+m^2)] \in L^{4-\epsilon}(\mathrm{d}^4 p), \epsilon > 0$, and hence $B_3 \in \mathscr{I}_{4-\epsilon}$. Letting

$$B_4 = \not{A}\frac{1}{p^2+m^2}, \qquad \text{(F15)}$$

we conclude by the same analysis that $B_4 \in \mathscr{I}_{4-\epsilon}$.

It is a general property of $\mathscr{I}_p$ spaces that $\|T\|_p \le \|T\|_{p'}$, $p \ge p'$. Thus, from (F13),

$$\left\|\not{A}S - \not{A}S_{m=0}\right\|_5 \le m^2 \|B_3\|_{4-\epsilon} + m\|B_4\|_{4-\epsilon}. \qquad \text{(F16)}$$

Referring again to (A8) Theorem A obtain

$$\left\|\not{A}S - \not{A}S_{m=0}\right\|_5 \leq (2\pi)^{\frac{4}{4-\epsilon}} \left\|\not{A}\right\|_{4-\epsilon} \left(m^2 \left\| \frac{\not{p}}{p^2(p^2+m^2)} \right\|_{4-\epsilon} + m \left\| \frac{1}{p^2+m^2} \right\|_{4-\epsilon} \right). \quad \text{(F17)}$$

The two $L^{4-\epsilon}(\mathrm{d}^4p)$ norms on the right-hand side of (F17) multiplied by m^2 and m both vanish as $m^{\epsilon/(4-\epsilon)}$ as $m \to 0$ when p is rescaled to mp.

This establishes the continuity of the $m = 0$ limit of $\det_5$ for any finite value of e by the Gohberg-Kreĭn theorem stated above.

Regarding Π_2 in (F1), we have already discussed off-shell renormalization in Sec.VIA. Subtracting off-shell adds the term (6.7) to $\mathrm{lndet}_{\mathrm{ren}}$. When this is combined with the right-hand side of (C7), which defines Π_2, the result is $\lim_{m=0} \Pi_2$ =finite.

Finally, the $m = 0$ limit of the photon-photon scattering graph Π_4 has been considered in detail for potentials with a $1/x$ fall off [51]. The conclusion is that $\lim_{m=0} \Pi_4$ =finite. The inclusion of potentials with a faster fall off such as those considered here can only reinforce this conclusion.

Summarizing, it has been established that $\lim_{m=0} \mathrm{lndet}_{\mathrm{ren}}$ =finite for off-shell charge renormalization and potentials $A_\mu \in \bigcap_{r \geq 4-\epsilon} L^r(\mathbb{R}^4)$. For zero mode supporting potentials the zero mass limit of $\mathrm{lndet}_{\mathrm{ren}}$ is not finite, but we know precisely where this divergence occurs, namely in $\det_3$.

[1] K.A. Olive, et al.(Particle Data Group), Chin.Phys. C, **38**, 090001(2014).
[2] S. Weinberg, *Quantum Theory of Fields* Vol. II: *Modern Applications* (Cambridge U. Press, Cambridge, UK, 1996).
[3] J. Schwinger, Phys. Rev. 82, 664 (1951).
[4] G. Schierholz (private communication commenting on [5]).
[5] M. Göckeler, R. Horsley, V. Linke, P. Rakow, G. Schierholz and H. Stüben, Phys. Rev. Lett. **80**, 4119 (1998).
[6] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators* (Springer, Berlin/Heidelberg, 1987).
[7] E. Seiler, in *Proceedings of the International Summer School of Theoretical Physics*, Poiana Brasov, Romania, 1981, edited by P. Dita, V. Georgescu, and R. Purice, Progress in Physics Vol. 5 (Birkhäuser, Boston, 1982), p. 263.
[8] B. Simon, *Trace Ideals and their Applications*, 2nd ed., Mathematical Surveys and Monographs Vol. 120 (American Mathematical Society, Providence, RI, 2005).
[9] N. Dunford and J. Schwartz, *Linear Operators Part II: Spectral Thoery* (Interscience, New York, 1963).
[10] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs Vol. 18 (American Mathematical Society, Providence, RI, 1969).
[11] B. Simon, Adv. Math. **24**, 244 (1977).
[12] B. Simon, Indiana Univ. Math. J. **26**, 1067 (1977).
[13] H. Hess, R. Schrader, and D. A. Ulenbrock, Duke Math. J. **44**, 893 (1977);
[14] B. Simon, J. Funct. Anal. **32**, 97 (1979);
[15] C. Vafa and E. Witten, Commun. Math. Phys. **95**, 257 (1984).
[16] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, Berlin, Heidelberg, 1999).
[17] V. S. Weisskopf, Kong. Dans. Vid. Selsk. Math-fys. Medd. **XIV**, No. 6 (1936).
[18] H. Araki and E.H. Lieb, Commun. Math. Phys. **18**, 160(1970).
[19] E. H. Lieb, Rev. Mod. Phys. **48**, 553 (1976).
[20] H. Hogreve, R. Schrader, and R. Seiler, Nucl. Phys. **B142**, 525 (1978).
[21] S. L. Adler, Phys. Rev. **D6**, 3445 (1972); **D7**, 3821(E)(1973).
[22] C. Itzykson, G. Parisi and J. B. Zuber, Phys. Rev. **D16**, 996 (1977); R. Balian, C. Itzykson, G. Parisi and J. B. Zuber, Phys. Rev. **D17**, 1041 (1978).
[23] E. B. Bogomol'nyĭ and Yu. A. Kubyshin, Sov. J. Nucl. Phys. **35**, 114 (1982); Yad. Fiz. **35**, 202 (1982); reprinted in *Large-Order Behaviour of Perturbation Theory*, edited by J. C. LeGuillou and J. Zinn-Justin, Current Physics-Sources and Comments Vol. 7 (North Holland, 1990), p.425.
[24] M. P. Fry, Phys. Rev. **D75**, 065002 (2007); *ibid* 069902(E) (2007). An extended version of this paper appears in hep-th/0612218.
[25] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966).
[26] J. Avron, I. Herbst and B. Simon, Duke Math. J. **45**, 847 (1978).
[27] L. S. Brown, R. D. Carlitz and C. Lee, Phys. Rev. **D16**, 417 (1977).
[28] R. Jackiw and C. Rebbi, Phys. Rev. **D16**, 1052 (1977).
[29] Y. Aharonov and A. Casher, Phys. Rev. **A19**, 2461 (1979).
[30] R. Musto, L. O'Raifeartaigh and A. Wipf, Phys. Lett. B **175**, 433 (1986).
[31] E.Seiler, Phys. Rev. **D22**, 2412 (1980).
[32] E. H. Lieb and M. Loss, *Analysis*, 2nd ed. (American Mathematical Society, Providence, RI, 2001).
[33] B. Simon, Phys. Rev. Lett. **36**, 1083 (1976).
[34] D. Hundertmark and B. Simon, J. Reine Angew. Math.

571, 107 (2004).
[35] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series*, Vol. 2: *Special Functions* (Gordon and Breach, New York, 1986).
[36] M. P. Fry, Phys. Rev. **D84**, 065021 (2011).
[37] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun, National Bureau of Standards (U. S. Government Printing Office, Washington, D.C., 1966).
[38] Digital Library of Mathematical Functions, National Inst. Standards and Tech., Ch.13 (dlmf.nist.gov).
[39] G. N. Watson, *Theory of Bessel Functions* (Cambridge U. Press, 1944).
[40] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).
[41] E. Seiler, Commun. Math. Phys. **42**, 163 (1975).
[42] T. Kato, Isr. J. Math. **13**, 135 (1972).
[43] B. Simon, Math. Z. **131**, 361 (1973).
[44] M. Schechter, J. Func. Anal. **20**, 93 (1975).
[45] D. Brydges, J. Fröhlich and E. Seiler, Ann. Phys. (N.Y.) **121**, 227 (1979).
[46] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).
[47] E. Seiler and B. Simon, Commun. Math. Phys., 45, 99 (1975).
[48] E. Seiler, *Lecture Notes in Physics* (Springer, Berlin/Heidelberg/New York, 1982), Vol. 159.
[49] S. L. Adler, Phys. Rev. D16, 2943 (1977).
[50] M. P. Fry, Phys. Rev. **D81**, 107701 (2010). An extended version of this paper appears in arXiv: 1005.4849.
[51] N. Ahmadiniaz, A. Huet, A. Raya and C. Schubert, Phys. Rev. **D87**, 125020 (2013).
[52] M. Reed and L. Rosen, Commun. Math. Phys. **36**, 123 (1974)
[53] P. Collela and O. Lanford, in *Constructive Quantum Field Theory*, edited by G. Velo and A. S. Wightman (Springer, Berlin/Heidelberg/New York, 1973), p.44.
[54] Z. Haba, Phy.Rev. **D29**, 1718(1984)
[55] M. Lüscher , hep-th/0102028
[56] L. D. Landau, in *Niels Bohr and the Development of Physics* (Pergamon Press, New York, 1955), p.52 and references therein.